Natural induction

(a model-theoretic study)

SINGLE-SIDED DRAFT

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10 Comparing inductive solutions

§ 10.1 Preliminaries

TODO: move each of the preliminary definitions to its respectively appropriate chapter, and restate here those that are appropriate to restate here

¶ 10.1.1 **Definitions [universal and partial closures]** For each formula $\varphi(\alpha_1,\ldots,\alpha_n)$ in any language:

(a) The universal closure of φ is:

$$\mathbf{C}\varphi := \forall\alpha_1, \ldots, \alpha_n\, \varphi.$$

(b) The partial closure of φ excepting α_i is:

$$\mathbf{C}_{-\alpha_i}\varphi := \forall\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\, \varphi.$$

¶ 10.1.2 **Definition [parameter-free induction axioms]** For each language L of arithmetic, for each each L-formula $\varphi(\alpha_1,\ldots,\alpha_n)$, and for each variable α, the parameter-free induction axiom for φ in α is:

$$\mathrm{I}(\varphi,\alpha) := (\mathbf{C}_{-\alpha}\varphi)[0/\alpha] \wedge \forall\alpha(\mathbf{C}_{-\alpha}\varphi \to (\mathbf{C}_{-\alpha}\varphi)[\alpha+1/\alpha]) \to \mathbf{C}\varphi.$$

¶ 10.1.3 **Definition [parameter-free induction schemes]** For each language L of arithmetic, the parameter-free induction scheme for L is:

$$\mathrm{I}(L) := \{\mathrm{I}(\varphi,\alpha) \ : \ \varphi(\alpha) \ L\text{-formula}\}.$$

¶ 10.1.4 **Definitions [inductive formulas]** For each language L of arithmetic and for each L-theory T:

(a) An L-formula $\varphi(\alpha)$ is inductive relative to T (alternatively: φ is T-inductive) if and only if:

$$T, \mathrm{I}(\varphi,\alpha) \vdash \forall\alpha\, \varphi(\alpha).$$

(b) An L-formula $\varphi(\alpha_1,\dots,\alpha_n)$ is inductive relative to T (alternatively: φ is T-inductive) if and only if $\mathbf{C}_{-\alpha}\varphi$ is for some α in $\{\alpha_1,\dots,\alpha_n\}$.

¶ 10.1.5 **Definition [inductive solutions]** For each language L of arithmetic and for each L-theory T: an L-formula ψ is an inductive solution for an L-formula φ relative to T (alternatively: ψ is a T-inductive solution for φ) if and only if:

– ψ is inductive relative to T; and

– $T \vdash \mathrm{C}\psi \to \mathrm{C}\varphi$.

§ 10.2 Introduction

¶ 10.2.1 The purpose of this short chapter is to motivate the terminology 'non-straightforward induction proof'. As remarked earlier, in actual mathematical practice the terminology used is seldom something like 'non-straightforward induction proof' but almost always something like:

– 'proof by strengthening one's induction hypothesis'; or

– '(inductive) proof by generalization (of the statement to be proved)'.

And indeed, when a working mathematician finds an arithmetic fact F seemingly requiring a non-straightforward proof it is almost always the case that:[*]

– F is sensibly modeled by the universal closure of a formula φ in some language L of arithmetic;

– the facts our working mathematician has at its disposal are sensibly modeled by an L-theory T of arithmetic;

– φ is non-inductive relative to T;

[*] I here disregard those cases where a non-straightforward proof of F proceeds by straightforward induction after one or more lemmas each has been proved by induction. Me disregarding these cases does not affect the point made, since many non-straightforward induction proofs do not proceed like that.

- a natural non-straightforward proof of F is sensibly modeled by a T-inductive solution ψ for φ;
- ψ is stronger than φ in an obvious sense—we typically have

$$T \vdash \mathbf{C}(\psi \to \varphi)$$
$$T \nvdash \mathbf{C}(\varphi \to \psi),$$

  or at least

$$T \vdash \mathbf{C}\psi \to \mathbf{C}\varphi$$
$$T \nvdash \mathbf{C}\varphi \to \mathbf{C}\psi.$$

While perhaps not representing everyday mathematical practice, there are nonetheless theories T of arithmetic with T-non-inductive formulas φ having T-inductive solutions ψ such that, in a precise and sensible sense, ψ is not stronger relative to T than φ. Thus a more general terminology—such as 'non-straightforward induction proof'—is motivated.

¶ 10.2.2 I present two facts in this chapter—one due to Hetzl and Wong (2018) and one due to Eric Johannesson (Johannesson and Lundstedt, 2026). Both facts show that there are theories T of arithmetic with T-non-inductive formulas φ having T-inductive solutions ψ such that, in a precise and sensible sense, ψ is not stronger relative to T than φ.

§ 10.3 Some ways to compare the logical strength of first-order formulas

¶ 10.3.1 In our manuscript in preparation, Eric Johannesson and I (2026) provide many ways to compare the strength of first-order formulas. For present purposes, the following definitions suffice.

¶ 10.3.2 **Definitions** For each language L, for each L-theory T, for each variable α, and for all L-formulas φ and ψ:

(a) φ is at least as T-strong in α as ψ, and ψ is at least as T-weak in α as φ, if and only if:

$$T \vdash \forall\alpha\colon\ \mathbf{C}_{-\alpha}\varphi \to \mathbf{C}_{-\alpha}\psi.$$

(b) φ is (strictly) T-stronger in α than ψ, and ψ is (strictly) T-weaker in α than φ, if and only if:

$$T \vdash \forall\alpha\colon\ \mathbf{C}_{-\alpha}\varphi \to \mathbf{C}_{-\alpha}\psi$$
$$T \nvdash \forall\alpha\colon\ \mathbf{C}_{-\alpha}\psi \to \mathbf{C}_{-\alpha}\varphi.$$

(c) φ and ψ are T-incomparable in α if and only if:

$$T \nvdash \forall\alpha\colon\ \mathbf{C}_{-\alpha}\varphi \to \mathbf{C}_{-\alpha}\psi$$
$$T \nvdash \forall\alpha\colon\ \mathbf{C}_{-\alpha}\psi \to \mathbf{C}_{-\alpha}\varphi.$$

¶ 10.3.3 **Remark** As should be obvious and expected, for each theory T and for each variable α:

'— is at least as T-strong in α as —'

and

'— is at least as T-weak in α as —'

are dual non-strict partial orders; and

'— is T-stronger in α than —'

and

'— is T-weaker in α than —'

are the respective correspondingly induced dual strict partial orders; and

'— and — are T-incomparable in α'

is the correspondingly induced incomparability relation.

§ 10.4 Inductive solutions that are not stronger than the formula they are an inductive solution for

¶ 10.4.1 The following fact is a straightforward generalization of Hetzl and Wong's (2018) Proposition 3.2.

¶ 10.4.2 **Fact** [Hetzl and Wong (2018)] For each language L of arithmetic and for each L-theory T of arithmetic such that $T \nvdash \mathrm{I}(L)$ there is an L-formula $\varphi(x)$ such that:

- φ has a T-inductive solution; and
- each T-inductive solution to φ is T-weaker in x than φ.

¶ 10.4.3 **Proof** § 10.A.

¶ 10.4.4 The following facts are due to Eric Johannesson. Definitions 4.2.2.2, of which Definition 4.2.2.2(a) is restated in § 10.B, define '$\mathcal{L}^{\mathrm{OR}}$' and '$\mathrm{PA}^{-}$', but for understanding the present relevance of Eric's discoveries, it suffices to know that $\mathcal{L}^{\mathrm{OR}}$ is a language of arithmetic and that PA^{-} is a theory of $\mathcal{L}^{\mathrm{OR}}$-arithmetic (Fact 4.2.2.4).

¶ 10.4.5 **Facts** [Johannesson and Lundstedt (2026)] There are $\mathcal{L}^{\mathrm{OR}}$-formulas $\varphi_1(x)$, $\varphi_2(x)$ and $\psi(x)$ such that:

(a) ψ is a PA^--inductive solution for φ_1, and ψ is PA^--weaker in x than φ_1.

(b) ψ is a PA^--inductive solution for φ_2, and ψ and φ_2 are PA^--incomparable in x.

¶ 10.4.6 **Proofs** § 10.B.

§ 10.A Hetzl and Wong's proof of Fact 10.4.2

¶ 10.A.1 **Fact 10.4.2 (restated)** For each language L of arithmetic and for each L-theory T of arithmetic such that $T \nvdash \mathrm{I}(L)$ there is an L-formula $\varphi(x)$ such that:

- φ has a T-inductive solution; and
- each T-inductive solution to φ is T-weaker in x than φ.

¶ 10.A.2 Just as the above fact is a straightforward generalization of Hetzl and Wong's (2018) Proposition 3.2, the following proof is a straightforward generalization of their proof of that result.

¶ 10.A.3 **Proof [of Fact 10.4.2]** Pick an L-sentence σ that is provable by $T + \mathrm{I}(L)$ but not by T (by assumption there are such sentences). Consider

$$\psi(x) \coloneqq \sigma \vee x \neq 0.$$

Clearly φ has a T-inductive solution, since $T + \mathrm{I}(L) \vdash \sigma$.

It remains to show that each T-inductive solution for φ is T-weaker in α than φ. We show the equivalent that there is no T-inductive solution for φ that is at least as T-strong in x as φ.

Suppose, towards a contradiction, that we have a $\psi(\alpha_1, ..., \alpha_n)$ such that: (1) ψ is a T-inductive solution for φ; and (2) ψ is at least as T-strong in x as φ.

By (1), at least one of the following holds:

$$T \vdash \mathbf{C}_{-\alpha_1}\psi(0, ..., \alpha_n)$$
$$\vdots$$
$$T \vdash \mathbf{C}_{-\alpha_n}\psi(\alpha_1, ..., 0).$$

Thus, whether x is in $\{\alpha_1, ..., \alpha_n\}$ or not:

$$T \vdash (\mathbf{C}_{-x}\psi)[0/x].$$

Thus by (2)—that is, by $T \vdash \forall x : \mathbf{C}_{-x}\psi \to \varphi$—we have

$$\text{(!)} \qquad T \vdash \varphi(0).$$

We have an L-model $M \models T \wedge \neg\sigma$ since $T \nvdash \sigma$ by assumption. Then $M \not\models \varphi(0)$, which contradicts (!).

§ 10.B Eric Johannesson's proofs of Facts 10.4.5

TODO

The proof uses results and definitions common to other stuff, so for purposes of harmony of terminology, definitions and results, to be fleshed it out together with that other stuff, when time for that other stuff comes. In the meantime, the manuscript of the following talk includes a rough proof in its § 5:

https://anderslundstedt.com/research/PhD-thesis/talks/lundstedt_talk_vienna_online_seminar_2021-05-12.pdf

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