Finitely non-standard models of Robinson arithmetic

Computational Logic Seminar

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September 2026

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References

- Lundstedt, Anders E.V. (2025): *Finitely non-standard models of Robinson arithmetic*; version 2025-06-10. In: *Natural induction (a model-theoretic study)*; PhD thesis in preparation; anderslundstedt.com/research/#induction-drafts
- Robinson, Raphael M. (1950): *An essentially undecidable axiom system*. In: *Proceedings of the International Congress of Mathematicians*, volume I; pages 729–730; American Mathematical Society; Providence 1952.

Introduction

- To the best of my knowledge, there is no systematic study of finitely non-standard models of Robinson arithmetic—that is, of models of Robinson arithmetic with a non-empty finite set of non-standard numbers.
- I (Lundstedt, 2025) have made a modest attempt at such a systematic study.
- My motivation for this study was my expectations of its applicability in my thesis research. (Those expectations were met.)
- Those were my motivations, but what I present today should hopefully have merit on its own.

Introduction

- While I have found no systematic study, the existing literature has examples of finitely non-standard models of Robinson arithmetic.
- I guess Robinson himself presented a finitely non-standard model of Robinson arithmetic in his address at the 1950 International Congress of Mathematicians.[1]

[1] No accompanying paper seems to have been published.

Introduction

The abstract for Robinson's (1950) address includes the (presumably) original axiomatization of Robinson arithmetic, and the following parenthetical:

> *(On the other hand, many simple formulas, such as* $0 + a = a$ *and* $a \leq a$*, are not provable from the given axioms.)*

One may easily show that $0 + a = a$ is not provable from (Robinson's original axiomatization of) Robinson arithmetic by exhibiting a suitable countermodel with two non-standard numbers.

Rough outline of main results

- The successor function is a permutation of the set of non-standard numbers (this follows trivially from the axioms).
- As is well-known, each permutation of a finite set has a unique decomposition into cycles on disjoint orbits.

Rough outline of main results

- The restriction of addition to

$$\{\langle a, n\rangle \ : \ a \text{ non-standard, } n \text{ standard}\}$$

is "tame", in the sense that this restriction is determined already by the permutation given by the successor function:

$$a + n$$

$$=$$

the result of starting at a and taking n steps in its cycle.

- The restriction of addition to

$$\{\langle \alpha, a\rangle \ : \ \alpha \text{ standard or non-standard, } a \text{ non-standard}\}$$

is "wild". Contrary to the previous restriction, this one is not determined by the successor function, but subject to some constraints—in particular, its range may only consist of non-standard numbers.

Rough outline of main results

- Similar to the case of addition, the restriction of multiplication to

$$\{\langle a, n\rangle \ : \ a \text{ non-standard, } n \text{ standard}\}$$

  is determined by the successor function and addition, whereas the restriction of multiplication to

$$\{\langle \alpha, a\rangle \ : \ \alpha \text{ standard or non-standard, } a \text{ non-standard}\}$$

  is not uniquely determined (by the successor function and addition), but subject to some constraints.

Preliminaries: Robinson arithmetic

- The language $\mathcal{L}^{\mathbf{Q}}$ of Robinson arithmetic is the first-order language with signature $\langle 0, S, +, \times \rangle$.
- The $\mathcal{L}^{\mathbf{Q}}$-theory Robinson arithmetic, notation '$\mathbf{Q}$', is axiomatized by:

(Q1) $$Sx \neq 0$$

(Q2) $$Sx = Sy \to x = y$$

(Q3) $$x = 0 \vee \exists y\, x = Sy$$

(Q4) $$x + 0 = x$$

(Q5) $$x + Sy = S(x + y)$$

(Q6) $$x \times 0 = 0$$

(Q7) $$x \times Sy = x \times y + x.$$

Preliminaries: Fragments of Robinson arithmetic

- The language $\mathcal{L}^{\mathrm{p}}$ is the $\mathcal{L}^{\mathbf{Q}}$-reduct with signature $\langle 0, S\rangle$.
- The progression fragment (of Robinson arithmetic), notation '$\mathbf{Q}^{\mathrm{p}}$', is the $\mathcal{L}^{\mathrm{p}}$-theory axiomatized by (Q1)–(Q3).
- The language $\mathcal{L}^{+}$ is the $\mathcal{L}^{\mathbf{Q}}$-reduct with signature $\langle 0, S, +\rangle$.
- The addition fragment (of Robinson arithmetic), notation '$\mathbf{Q}^{+}$', is the $\mathcal{L}^{+}$-theory axiomatized by (Q1)–(Q5).

Preliminaries: F.n.s. models

- For $L = \mathcal{L}^{\mathrm{P}}$, $L = \mathcal{L}^{+}$ and $L = \mathcal{L}^{\mathbf{Q}}$, an L-model is finitely non-standard if and only if:
  - its domain is $\mathbb{N} + A$ for some finite non-empty set A of non-standard numbers disjoint from $\mathbb{N}$; and
  - restricting the domain to $\mathbb{N}$ is possible and this restriction is the standard L-model.
- 'f.n.s.' abbreviates 'finitely non-standard'.
- When I use 'f.n.s. model'—without specifying a language—what I mean is an f.n.s. model of any of the three languages $\mathcal{L}^{\mathrm{P}}$, $\mathcal{L}^{+}$ and $\mathcal{L}^{\mathbf{Q}}$, to the respective extent a model of each language makes sense in the given context. This convention also applies, mutatis mutandis, in similar cases where no language is specified.

F.n.s. models of the progression fragment: Fact 1

- Recall the axiomatization of $\mathbf{Q}^{\mathrm{p}}$:

(Q1)	$Sx \neq 0$
(Q2)	$Sx = Sy \rightarrow x = y$
(Q3)	$x = 0 \vee \exists y\, x = Sy$

- FACT 1. An f.n.s. model is a model of $\mathbf{Q}^{\mathrm{p}}$ if and only if S restricted to the set of non-standard numbers is a permutation of the set of non-standard numbers.

F.n.s. models of the progression fragment: Proof of Fact 1

- FACT 1. An f.n.s. model is a model of $\mathbf{Q}^{\mathrm{p}}$ if and only if S restricted to the set of non-standard numbers is a permutation of the set of non-standard numbers.
- PROOF.
  - The if direction is trivial.
  - For the only if direction, take any f.n.s. model that is a model of (Q1)–(Q3).
  - By (Q1) and (Q2) and by the definition of '— is an f.n.s. model', none of our model's non-standard numbers has a standard successor.
  - Thus since 0 is standard:
    - By (Q3) each of our non-standard numbers has a non-standard predecessor.
    - Thus (Q2) and (Q3) give that S restricted to the set of non-standard numbers is a permutation of the set of non-standard numbers.

Some conveniences

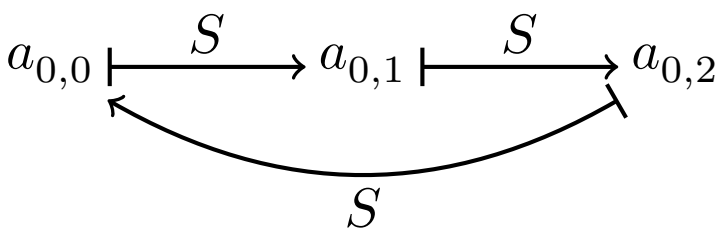
- N_{p} is an arbitrarily chosen f.n.s. $\mathcal{L}^{\mathrm{p}}$-model of $\mathbf{Q}^{\mathrm{p}}$.
- We denote the set of non-standard numbers of N_{p} by 'A'.

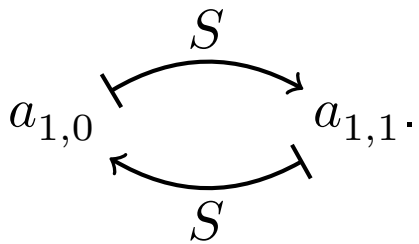
Some conveniences

- As is well-known, each permutation of a finite set has a unique decomposition into cycles on disjoint orbits. Thus by Fact 1 there is a unique such decomposition of A corresponding to its S-permutation.
- EXAMPLE.

$$A = A_0 + A_1 \quad A_0 = \{a_{0,0},\ a_{0,1},\ a_{0,2}\} \quad A_1 = \{a_{1,0},\ a_{1,1}\}.$$

$S \downarrow A$:





Some conveniences

- We use the following generalization of the notation in the previous example.
- By 'ν', we denote the number of cycles in the decomposition.
- We use '$A_$', with indices denoting natural numbers, to denote the cycles—that is, the ν cycles are:

$$A_0,\ ...,\ A_{\nu-1}.$$

- A cycle index is a natural number i such that $0 \leq i < \nu$.
- We use '$\mu[-]$' to denote the lengths of the cycles—that is, for each cycle index i:

$$\mu[i] = \text{the length of cycle } A_i = \text{the size of } A_i.$$

- We use

$$`a_{-,0}',\ `a_{-,1}',\ `a_{-,2}',\ ...$$

to denote the (non-standard) numbers in a cycle—that is, for each cycle index i, the $\mu[i]$ non-standard numbers in A_i are:

$$a_{i,0}, ...,\ a_{i,\mu[i]-1}.$$

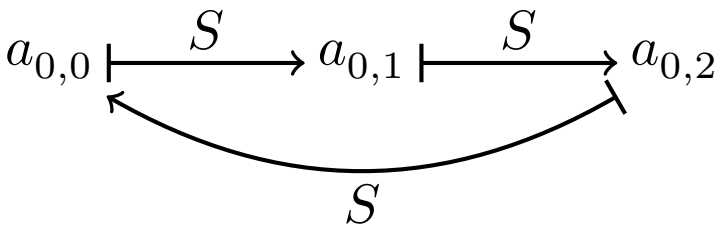
Some conveniences

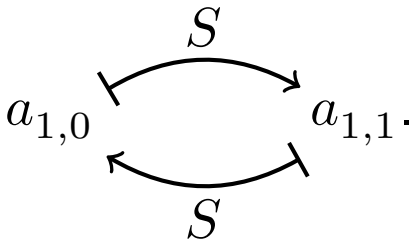
ASSUMPTION. Without loss of generality, for each cycle index i:

$$Sa_{i,j} = a_{i,j+1} \quad \text{if} \quad j < \mu[i] - 1$$
$$Sa_{i,\mu[i]-1} = a_{i,0}.$$

Some conveniences

EXAMPLE. We revisit our previous example.





- There are $\nu = 2$ cycles.
- The cycle indices are 0 and 1.
- There are $\mu[0] = 3$ elements in the cycle with cycle index 0.
- There are $\mu[1] = 2$ elements in the cycle with cycle index 1.
- The assumption on the previous slide is satisfied.

Some conveniences: The function a_i

DEFINITION. For each cycle index i:

$$a_i : \quad \mathbb{Z} \to A_i$$
$$a_i(j) := a_{i,k} \quad \text{if and only if} \quad j \equiv k \mod \mu[i].$$

Some conveniences: An example

EXAMPLE. Suppose $\mu[0] = 3$—that is:

$$A_0 = \{a_{0,0},\ a_{0,1},\ a_{0,2}\}.$$

We then have:

$$\begin{aligned} a_0(-1) &= a_{0,2} \\ a_0(-2) &= a_{0,1} \\ a_0(-3) &= a_{0,0} \\ a_0(-4) &= a_{0,2} \\ a_0(-5) &= a_{0,1} \\ &\vdots \end{aligned} \qquad \begin{aligned} a_0(0) &= a_{0,0} \\ a_0(1) &= a_{0,1} \\ a_0(2) &= a_{0,2} \\ a_0(3) &= a_{0,0} \\ a_0(4) &= a_{0,1} \\ &\vdots \end{aligned}$$

F.n.s. models of the addition fragment: Recall $\mathbf{Q}^+$

Recall the axiomatization of $\mathbf{Q}^+$:

(Q1) $$Sx \neq 0$$

(Q2) $$Sx = Sy \to x = y$$

(Q3) $$x = 0 \vee \exists y\, x = Sy$$

(Q4) $$x + 0 = x$$

(Q5) $$x + Sy = S(x + y)$$

F.n.s. models of the addition fragment: Fact 2

- ASSUMPTION. N_+ is an arbitrarily chosen f.n.s. $\mathcal{L}^+$-expansion of N_{p}.
- FACT 2. N_+ is a model of $\mathbf{Q}^+$ if and only if:
  - For each non-standard a and for each standard n: $a + n = S^n a$.
  - For each (standard or non-standard) α, for each cycle index i, and for each integer j:

    (F2$_j$) $$\alpha + a_i(j) = S^j(\alpha + a_{i,0}).$$

F.n.s. models of the addition fragment: Proof Fact 2

- For each non-standard a and for each standard n: $a + n = S^n a$.
- PROOF.

$$\begin{aligned}
a + 0 &= a \\
&= S^0 a \\
a + 1 &= a + S0 \\
&= Sa \\
&= S^1 a \\
a + 2 &= a + S1 \\
&= S(a + 1) \\
&= SS^1 a \\
&= S^2 a \\
&\vdots
\end{aligned}$$

F.n.s. models of the addition fragment: Proof of Fact 2

- For each (standard or non-standard) α, for each cycle index i, and for each integer j:

  $(\mathsf{F2}_j)$ $\qquad\qquad \alpha + a_i(j) = S^j(\alpha + a_{i,0}).$

- PROOF FOR $0 \leq j < \mu[i]$. We show the equivalence between $\alpha + Sa_{i,j} = S(\alpha + a_{i,j})$ and $(\mathsf{F2})_{j+1}$ ($(\mathsf{F2})_0$ holds trivially).

$$\alpha + Sa_{i,0} = S(\alpha + a_{i,0})$$

iff

$$\alpha + a_i(1) = S^1(\alpha + a_{i,0})$$

$$\alpha + Sa_{i,1} = S(\alpha + a_{i,1})$$

iff

$$\alpha + a_i(2) = SS^1(\alpha + a_{i,0}) \quad \text{(by previous)}$$

iff

$$\alpha + a_i(2) = S^2(\alpha + a_{i,0})$$

F.n.s. models of the addition fragment: Proof of Fact 2

$$\alpha + Sa_{i,2} = S(\alpha + a_{i,2})$$

iff

$$\alpha + a_i(3) = SS^2(\alpha + a_{i,0}) \quad \text{(by previous)}$$

iff

$$\alpha + a_i(3) = S^3(\alpha + a_{i,0})$$

$$\vdots$$

F.n.s. models of the addition fragment: A corollary of Fact 2

- COROLLARY. Suppose $N_+ \vDash \mathbf{Q}^+$. Then N_+ is uniquely determined by its $\mathcal{L}^{\mathrm{p}}$-reduct together with the restriction of $+$ to

$$\{\langle \alpha, a_{i,0} \rangle \ : \ \alpha \text{ standard or non-standard, } i \text{ cycle index}\}.$$

- PROOF. We need to show that this uniquely determines $+$. The definition of '— is an f.n.s. $\mathcal{L}^+$-model' determines $+$ on the standard numbers. Given the $\mathcal{L}^{\mathrm{p}}$-reduct and the given restriction of $+$, the remaining cases are determined by the equations in Fact 2.

F.n.s. models of the addition fragment: Another corollary of Fact 2

COROLLARY. Suppose $N_+ \vDash \mathbf{Q}^+$. Then for each cycle index i and for each (standard or non-standard) α there is a cycle index k such that:

- $\alpha + a$ is in A_k for each a in A_i; and
- $\mu[k]$ divides $\mu[i]$.

F.n.s. models of Robinson arithmetic: Recall **Q**

Recall the axiomatization of **Q**:

(Q1) $$Sx \neq 0$$

(Q2) $$Sx = Sy \to x = y$$

(Q3) $$x = 0 \vee \exists y\, x = Sy$$

(Q4) $$x + 0 = x$$

(Q5) $$x + Sy = S(x + y)$$

(Q6) $$x \times 0 = 0$$

(Q7) $$x \times Sy = x \times y + x.$$

F.n.s. models of Robinson arithmetic: Expanding N_+

ASSUMPTION. N is an arbitrarily chosen f.n.s. $\mathcal{L}^{\mathbf{Q}}$-expansion of N_+ that models $\mathbf{Q}^+$.

F.n.s. models of Robinson arithmetic: Addition to the right

- DEFINITION. Addition to the right, notation '$\oplus_$', is defined for each number β in N:

$$\oplus_\beta : \quad N \to N$$
$$\oplus_\beta \alpha := \alpha + \beta.$$

- EXAMPLE. For all α and β in N:

$$\oplus^0_\beta \alpha = \alpha$$
$$\oplus^1_\beta \alpha = \alpha + \beta$$
$$\oplus^2_\beta \alpha = (\alpha + \beta) + \beta$$
$$\oplus^3_\beta \alpha = ((\alpha + \beta) + \beta) + \beta$$
$$\vdots$$

F.n.s. models of Robinson arithmetic: Fact 3

FACT 3. N is a model of **Q** if and only if:

- For each non-standard a and for each standard n:

$$a \times n = \oplus_a^n 0.$$

- For each standard n, for each cycle index i, and for each integer j:

$$n \times a_i(j) = S^{n \times j}(n \times a_{i,0}).$$

- For each non-standard a, for each cycle index i, and for each non-negative integer j:

$$a \times a_i(j) = \oplus_a^j (a \times a_{i,0}).$$