NON-STRAIGHTFORWARD INDUCTION PROOFS AND THE COMPARATIVE STRENGTH OF INDUCTIVE SOLUTIONS

Presentation at the online seminar of the Computational Logic group of the Institute of Discrete Mathematics and Geometry, TU Wien

Anders Lundstedt
Department of philosophy, Stockholm university
anders.lundstedt@philosophy.su.se
anderslundstedt.com

May 12, 2021

Contents

§ 1 Non-straightforward induction proofs

¶ 1.1 To get everybody on the same page, let us look at two examples demonstrating the seeming need for what I have chosen to call 'non-straightforward induction proofs'.

¶ 1.2 For all natural numbers n:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{(n+1)^2} < 2.$$

¶ 1.3 With recursion instead of ellipsis, we have the following.

¶ 1.4 DEFINITION

$$\begin{aligned} \Sigma_1 &: \mathbb{N} \longrightarrow \mathbb{Q} \\ \Sigma_1(0) &:= 1 \\ \Sigma_1(n+1) &:= \Sigma_1(n)+(n+2)^{-2} \end{aligned}$$

¶ 1.5 FACT For all natural numbers n:

$$\Sigma_1(n) < 2.$$

¶ 1.6 PROOF ATTEMPT Straightforward induction:

- Base case:

$$\begin{aligned} \Sigma_1(0) &= 1 \quad \text{(by definition)} \\ &< 2. \end{aligned}$$

- Induction step:

$$\begin{aligned} \Sigma_1(n+1) &= \Sigma_1(n)+(n+2)^{-2} \quad \text{(by definition)} \\ &< 2+(n+2)^{-2} \quad \text{(by induction hypothesis).} \end{aligned}$$

We have $2+(n+2)^{-2} \nless 2$ for all n—so how do we proceed from here? It seems like this proof attempt is "stuck".

¶ 1.7 We abort the proof attempt. Instead we prove the following stronger fact. (In what sense is it stronger?)

¶ 1.8 FACT For all natural numbers n: $\Sigma_1(n) \leq 2-(n+1)^{-1}$.

¶ 1.9 PROOF Straightforward induction:

- Base case:

$$\begin{aligned} \Sigma_1(0) &= 1 && \text{(by definition)} \\ &\leq 1 \\ &= 2-(0+1)^{-1}. \end{aligned}$$

- Induction step:

$$\begin{aligned} &\Sigma_1(n+1) \\ &\quad = \Sigma_1(n)+(n+2)^{-2} && \text{(by definition)} \\ &\quad \leq 2-(n+1)^{-1}+(n+2)^{-2} && \text{(by induction hypothesis)} \\ &\quad = 2-(n+2)^{-1}-(n+1)^{-1}(n+2)^{-2} && \text{(by some arithmetic)} \\ &\quad \leq 2-(n+2)^{-1} \\ &\quad = 2-((n+1)+1)^{-1}. \end{aligned}$$

□

¶ 1.10 We have that the sum of any initial segment of the odd natural numbers is a perfect square:

$$0 = 0^2,$$

$$1 = 1^2,$$

$$1+3 = 4 = 2^2,$$

$$1+3+5 = 9 = 3^2,$$

$$1+3+5+7 = 16 = 4^2,$$

$$\vdots$$

¶ 1.11 DEFINITION

$$\begin{aligned} \Sigma_2 &: \mathbb{N} \longrightarrow \mathbb{N} \\ \Sigma_2(0) &:= 0 \\ \Sigma_2(n+1) &:= \Sigma_2(n)+2n+1 \end{aligned}$$

¶ 1.12 FACT For all natural numbers n there is a natural number k such that $\Sigma_2(n) = k^2$.

¶ 1.13 PROOF ATTEMPT Straightforward induction:

– Base case:

$$\begin{aligned} \Sigma_2(0) &= 0 \quad \text{(by definition)} \\ &= 0^2. \end{aligned}$$

Take $k = 0$.

– Induction step:

$$\begin{aligned} &\Sigma_2(n+1) \\ &\quad = \Sigma_2(n)+2n+1 \quad \text{(by definition)} \\ &\quad = k^2+2n+1 \quad \text{(for some } k\text{, by induction hypothesis).} \end{aligned}$$

It is not true that k^2+2n+1 is a perfect square for all k and n—so how do we proceed from here? It seems like this proof attempt is “stuck”.

¶ 1.14 We abort the proof attempt. Instead we prove the following stronger fact. (In what sense is it stronger?)

¶ 1.15 FACT For all natural numbers n: $\Sigma_2(n) = n^2$.

¶ 1.16 PROOF Straightforward induction:

- Base case:

$$\begin{aligned}\Sigma_2(0) &= 0 && \text{(by definition)}\\ &= 0^2.\end{aligned}$$

- Induction step:

$$\begin{aligned}\Sigma_2(n+1) &= \Sigma_2(n)+2n+1 && \text{(by definition)}\\ &= n^2+2n+1 && \text{(by induction hypothesis)}\\ &= (n+1)^2.\end{aligned}$$

□

¶ 1.17 — Let F be a fact represented by a sentence $\forall x.\varphi(x)$ in a suitable language of arithmetic.

— A proof of F is a *straightforward induction proof* if it can be represented by a proof tree of the form

$$\dfrac{\begin{array}{c}\vdots\\ \varphi(0) \wedge \forall x(\varphi(x) \longrightarrow \varphi(x+1))\end{array}}{\forall x.\varphi(x)} \quad .$$

— A proof of F is a *non-straightforward induction proof* if it can be represented by a proof tree of the form

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \psi(0) \wedge \forall x(\psi(x) \longrightarrow \psi(x+1))\end{array}}{\forall x.\psi(x)}\\ \vdots\\ \forall x.\varphi(x)\end{array} \quad .$$

¶ 1.18 — Simply putting the “non-straightforward induction proof tree form” above the “straightforward induction proof tree form” gives a construction turning any representation of a non-straightforward induction proof into a representation of a straightforward induction proof:

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \psi(0) \wedge \forall x(\psi(x) \longrightarrow \psi(x+1))\end{array}}{\forall x.\psi(x)}\\ \vdots\\ \forall x.\varphi(x)\\ \vdots\\ \dfrac{\varphi(0) \wedge \forall x(\varphi(x) \longrightarrow \varphi(x+1))}{\forall x.\varphi(x)}\end{array} \quad .$$

— Thus these proof tree forms will not, on their own, be able to tell us whether a fact with a non-straightforward induction proof has no straightforward induction proof.

— One could of course argue that the above construction shows that all facts with non-straightforward induction proofs do have straightforward induction proofs. This just changes the problem to that of deciding whether facts with non-straightforward induction proofs have “natural” straightforward induction proofs.

§ 2 An example

¶ 2.1 — Let L be the first-order language $\langle 0,1,+,\times,\Sigma\rangle$.

— The reduct $\mathcal{L}_0 := \langle 0,1,+,\times\rangle$ has the usual standard model (the one with domain the natural numbers).

— *True $\mathcal{L}_0$-arithmetic*, notation ‘Tr_0’, is the theory of this standard model.

— For an L-formula $\varphi(x)$ the corresponding *parameter-free (successor) induction axiom*, notation ‘$\mathrm{I}\varphi$’, is the L-sentence:

$$\varphi(0) \wedge \forall x(\varphi(x) \longrightarrow \varphi(x+1)) \longrightarrow \forall x.\varphi(x).$$

¶ 2.2 — Using L we represent Definition 1.11—the definition of Σ_2 as the sums of initial segments of the odd natural numbers—by the L-sentences:

$$\mathrm{DEF}_0 :\equiv \Sigma(0) = 1,$$

$$\mathrm{DEF}_+ :\equiv \forall x\colon \Sigma(x+1) = \Sigma(x)+2x+1.$$

(Bracketing of L-terms will not matter.)

— We represent Fact 1.12—the fact that $\Sigma_2(n)$ is a perfect square for all n—by the L-sentence $\forall x.\varphi(x)$, with

$$\varphi(x) :\equiv \exists y\colon \Sigma(x) = y\times y.$$

— We represent Fact 1.15—the strengthening of Fact 1.12—by the L-sentence $\forall x.\psi(x)$, with

$$\psi(x) :\equiv \Sigma(x) = x\times x.$$

¶ 2.3 FACT [Lundstedt (2021)] Let

$$T :\equiv \mathrm{Tr}_0+\mathrm{DEF}_0+\mathrm{DEF}_+.$$

Then:

(1) $T \vdash \varphi(0)$;

(2) $T \nvdash \forall x\colon \varphi(x) \longrightarrow \varphi(x+1)$;

(3) $T \vdash \psi(0)$;

(4) $T \vdash \forall x\colon \psi(x) \longrightarrow \psi(x+1)$; and

(5) $T \vdash \forall x.\psi(x) \longrightarrow \forall x.\varphi(x)$.

¶ 2.4

- (2) is a non-inductiveness result: $\varphi(x)$ is *not T-inductive*.
- Similarly, (3) together with (4) is an inductiveness result: $\psi(x)$ is *T-inductive*.
- Furthermore, adding (5) to (3) and (4) shows that $\psi(x)$ is a *T-inductive solution* for $\varphi(x)$.

¶ 2.5

- I will not present a proof of Fact 2.3 here.
- All of (1)–(5) but the non-inductiveness result (2) are easy.
- (2) is equivalent to: (×) there is a non-standard L-model $M \vDash T$ with a non-standard number c such that

  $$M \vDash \varphi(c) \wedge \neg\varphi(c+1),$$

  where

  - an L-model is *non-standard* if and only if it is not an expansion of the standard $\mathcal{L}_0$-model; and
  - an element of an L-model is *non-standard* if and only if it is not the interpretation of a numeral.
- One proof of (×) goes by cleverly expanding an arbitrary non-standard $\mathcal{L}_0$-model of Tr_0 to an L-model of T—the required cleverness consisting in interpreting Σ such that $\varphi(c) \wedge \neg\varphi(c+1)$ holds for some non-standard number c in the expansion.
- Another proof of (×) goes by applying the compactness theorem after some non-trivial preparation.

¶ 2.6 – Fact 2.3 provides the following precise sense in which Fact 1.12 cannot be naturally proved by straightforward induction.

– Suppose a working mathematician has at their disposal:

  – any set of arithmetical facts represented by a set T' of L-sentences such that

$$T \vdash T' \vdash \text{Robinson arithmetic} + \text{DEF}_0 + \text{DEF}_+;$$

  – a (limited) principle of mathematical induction on natural numbers, as represented by the schema

$$\{\mathrm{I}\varphi \mid \varphi(x) \text{ any } L\text{-formula}\};$$

  – reasoning that can be represented by first-order derivations of L-sentences from L-sentences.

– Then our working mathematician can prove Fact 1.12 with one application of induction, but that application of induction cannot be induction on n in '$\Sigma_2(n)$ is a perfect square'.

¶ 2.7 – I think the above is a sensible sense in which Fact 1.12 cannot be naturally proved by straightforward induction.

– It is probably not the most sensible such sense possible.

  – In the business of modeling, we need to idealize.

  – The main idealization in this case is our modeling of the working mathematician trying to prove Fact 1.12 as trying to construct a first-order derivation of an L-sentence from some set of L-sentences.

  – Thus our model does not take into account that our working mathematician probably has at their disposal reasoning not represented by such derivations, as well as facts not represented by L-sentences.

– Most likely, one could come up with other models and corresponding results that more faithfully captures that Fact 1.12 has no natural straightforward induction proof.

– Still, I think we have captured much of the situation we (as working mathematicians) were in when trying to prove Fact 1.12. In particular, just after its definition, we had established nothing specific to the function Σ_2 other than its two defining equations, which is reflected in our model by the fact that 'Σ' occurs in DEF_0 and DEF_+ but in no other sentences in T.

¶ 2.8 We have the following strength comparisons of $\varphi(x)$ and $\psi(x)$.

- $\psi(x)$ is *logically stronger* than $\varphi(x)$:

  $$\vdash \forall x\colon \psi(x) \longrightarrow \varphi(x),$$

  $$\nvdash \forall x\colon \varphi(x) \longrightarrow \psi(x).$$

- $\psi(x)$ is *T-stronger* than $\varphi(x)$:

  $$T \vdash \forall x\colon \psi(x) \longrightarrow \varphi(x),$$

  $$T \nvdash \forall x\colon \varphi(x) \longrightarrow \psi(x).$$

- $\forall x.\psi(x)$ is *logically stronger* than $\forall x.\varphi(x)$:

  $$\vdash \forall x.\psi(x) \longrightarrow \forall x.\varphi(x),$$

  $$\nvdash \forall x.\varphi(x) \longrightarrow \forall x.\psi(x).$$

- $\forall x.\psi(x)$ is *at least as T-strong* as $\forall x.\varphi(x)$:

  $$T \vdash \forall x.\psi(x) \longrightarrow \forall x.\varphi(x).$$

  (I have not put any effort into checking whether $\forall x.\psi(x)$ is *T-stronger* than $\forall x.\varphi(x)$—that is, whether we also have $T \nvdash \forall x.\varphi(x) \longrightarrow \forall x.\psi(x)$.)

¶ 2.9

- Consider, as exemplified by (1)–(5), a theory T and a not T-inductive formula $\varphi(x)$ with a T-inductive solution $\psi(x)$.

- The rest of this talk will focus on how such $\varphi(x)$ and $\psi(x)$ may compare to each other in terms of various relations of strength.

§ 3 The theory of the non-negative parts of discretely ordered commutative rings

¶ 3.1

- From here on we work with the *first-order language of ordered rings*:

  $$\mathcal{L}^{\mathsf{OR}} := \langle 0,1,+,\times,<\rangle.$$

- Just as its reduct $\mathcal{L}_0$ (from § 2), this language too has its expected standard model with domain the natural numbers.

- *True $\mathcal{L}^{\mathsf{OR}}$-arithmetic*, notation 'Tr', is the theory of this standard model.

- An $\mathcal{L}^{\mathsf{OR}}$-theory T is *sound* if and only if $\mathrm{Tr} \vdash T$ (the intention with this terminology being that——in the present context——"sound theories" should not contradict arithmetic).

- For an $\mathcal{L}^{\mathsf{OR}}$-formula $\varphi(x)$ the corresponding *parameter-free (successor) induction axiom*, notation 'Iφ', is the $\mathcal{L}^{\mathsf{OR}}$-sentence:

  $$\varphi(0) \wedge \forall x(\varphi(x) \longrightarrow \varphi(x+1)) \longrightarrow \forall x.\varphi(x).$$

- The *parameter-free (successor) induction scheme* for $\mathcal{L}^{\mathsf{OR}}$, notation 'I$\mathcal{L}^{\mathsf{OR}}$', is the set of $\mathcal{L}^{\mathsf{OR}}$-formulas:

  $$\{\mathrm{I}\varphi \mid \varphi(x) \text{ any } \mathcal{L}^{\mathsf{OR}}\text{-formula}\}.$$

¶ 3.2 DEFINITION The *theory of the non-negative parts of discretely ordered commutative rings*, notation 'PA^-', is the $\mathcal{L}^{\mathsf{OR}}$-theory with the following axioms.

- Associativity and commutativity of addition and multiplication, and distributivity of multiplication over addition:

  $$\forall x,y,z\colon (x+y)+z = x+(y+z),$$

  $$\forall x,y\colon x+y = y+x,$$

  $$\forall x,y,z\colon (xy)z = x(yz),$$

  $$\forall x,y\colon xy = yx,$$

  $$\forall x,y,z\colon x(y+z) = xy+xz.$$

- 0 is an additive identity and a multiplicative zero, and 1 is a multiplicative identity:

  $$\forall x\colon x+0 = x,$$

  $$\forall x\colon x \times 0 = 0,$$

  $$\forall x\colon x \times 1 = x.$$

- < is a strict total order respected by + and ×:

  $$\forall x,y,z\colon x < y \wedge y < z \longrightarrow x < z,$$

  $$\forall x\colon x \nless x,$$

  $$\forall x,y\colon x < y \vee x = y \vee x > y,$$

  $$\forall x,y,z\colon x < y \longrightarrow x+z < y+z,$$

  $$\forall x,y,z\colon z > 0 \wedge x < y \longrightarrow xz < yz.$$

- Smaller elements can be subtracted from larger elements:

  $$\forall x,y\colon x < y \longrightarrow \exists z\ y = x+z.$$

- The order is discrete:

  $$0 < 1 \wedge \forall x(x > 0 \longrightarrow x \geq 1).$$

- 0 is the least element:

  $$\forall x\colon 0 \leq x.$$

¶ 3.3 REMARK

- Peano arithmetic, notation ‘PA’, may be defined by

  $$\mathrm{PA} :\equiv \mathrm{PA}^{-}+\mathrm{I}\mathcal{L}^{\mathbf{OR}}.$$

- We have the strict provabilities

  $$\mathrm{PA} \vdash \mathrm{PA}^{-} \vdash \text{Robinson arithmetic}.$$

¶ 3.4 — A model of PA^- is *standard* if it is isomorphic to the standard $\mathcal{L}^{OR}$-model; otherwise it is *non-standard*.

— An element of a model of PA^- is a *standard number* if it is the interpretation of a numeral; otherwise it is a *non-standard number*.

— It follows that a model of PA^- is non-standard if and only if it contains non-standard numbers.

§ 4 Inductive formulas; non-trivially not inductive formulas; inductive solutions

¶ 4.1 DEFINITION [A straightforward generalization of Hetzl and Wong's (2018) *inductive formula*]

– Let T be an $\mathcal{L}^{\mathbf{OR}}$-theory and let $\varphi(x)$ be an $\mathcal{L}^{\mathbf{OR}}$-formula.

– $\varphi(x)$ is *T-inductive* if and only if

$$T \vdash \varphi(0) \land \forall x(\varphi(x) \longrightarrow \varphi(x+1)).$$

– If $\varphi(x)$ is not T-inductive then $\varphi(x)$ is *non-trivially* not T-inductive if and only if

$$T+\mathrm{I}\mathcal{L}^{\mathbf{OR}} \vdash \forall x.\varphi(x).$$

¶ 4.2 – Suppose T is an $\mathcal{L}^{\mathbf{OR}}$-theory and suppose $\varphi(x)$ is not T-inductive:

$$T \nvdash \varphi(0) \land \forall x(\varphi(x) \longrightarrow \varphi(x+1)).$$

– This is equivalent to

$$T+\mathrm{I}\varphi \nvdash \forall x.\varphi(x),$$

which models that the fact represented by $\forall x.\varphi(x)$ is unprovable by straightforward induction from the facts represented by T.

– Suppose furthermore that $\varphi(x)$ is non-trivially not T-inductive——that is, in addition to the above, T together with one or more induction axioms do prove $\forall x.\varphi(x)$.

– Since the conjunction of any two induction axioms is logically equivalent to a third induction axiom——this is referred to as a “folklore result” by Hetzl and Wong (2018)——that $\varphi(x)$ is non-trivially not T-inductive is equivalent to that there is a $\psi(x)$ such that

$$T+\mathrm{I}\psi \vdash \forall x.\varphi(x).$$

– This is equivalent to that there is T-inductive formula $\psi(x)$ such that

$$T \vdash \forall x.\psi(x) \longrightarrow \forall x.\varphi(x).$$

This paragraph motivates the following definition.

¶ 4.3 DEFINITION

- Let T be an $\mathcal{L}^{\mathbf{OR}}$-theory and let $\varphi(x)$ and $\psi(x)$ be $\mathcal{L}^{\mathbf{OR}}$-formulas.
- $\psi(x)$ is a *T-inductive solution* for $\varphi(x)$ if and only if:
  - $\psi(x)$ is T-inductive; and
  - $T \vdash \forall x.\psi(x) \longrightarrow \forall x.\varphi(x)$.

§ 5 Some results and open questions

¶ 5.1 — Let T be a sound extension of PA^-.

— Let $\varphi(x)$ be not T-inductive.

— Let $\psi(x)$ be a T-inductive solution for $\varphi(x)$.

— When T, $\varphi(x)$ and $\psi(x)$ come from the modeling of a non-straightforward induction proof found in the wild it is often (almost always?) the case that we have

$$T \vdash \forall x\colon \psi(x) \longrightarrow \varphi(x).$$

In theory this need not be the case, as the following result shows.

¶ 5.2 FACT [A straightforward generalization of Hetzl and Wong's (2018) Proposition 3.2]

— Let T be a sound extension of PA^- such that $T \nvdash I\mathcal{L}^{OR}$.

— Then there is a non-trivially not T-inductive $\varphi(x)$ for which there is no T-inductive solution $\psi(x)$ such that

$$T \vdash \forall x\colon \psi(x) \longrightarrow \varphi(x).$$

¶ 5.3 PROOF The following is a straightforward generalization of the corresponding proof by Hetzl and Wong.

— Pick an $\mathcal{L}^{OR}$-sentence σ such that $T+I\mathcal{L}^{OR} \vdash \sigma$ and $T \nvdash \sigma$.

— Define

$$\varphi(x) :\equiv \sigma \lor x \neq 0.$$

— Take a model $M \vDash T+\neg\sigma$ (such a model exists since $T \nvdash \sigma$). Note:

(×) $M \nvDash \varphi(0)$.

Thus $T \nvdash \varphi(0)$, so $\varphi(x)$ is not T-inductive.

— Since clearly $T+I\mathcal{L}^{OR} \vdash \forall x.\varphi(x)$, $\varphi(x)$ is non-trivially not T-inductive.

- Next take any T-inductive solution for $\psi(x)$. We have, since $\psi(x)$ is T-inductive,

  $$T \vdash \psi(0).$$

  Thus if

  $$T \vdash \forall x\colon \psi(x) \longrightarrow \varphi(x),$$

  we would have

  $$T \vdash \varphi(0),$$

  contradicting (×). □

¶ 5.4

- Note that in Proof 5.3, $T \nvdash \varphi(0)$ is the reason that $\varphi(x)$ is not T-inductive.
- This does not match mathematical practice, where the reason for a straightforward induction attempt breaking down is virtually always the unprovability of the induction step.
- We could instead have used, for example, the definition

  $$\varphi(x) :\equiv \sigma \vee \mathrm{Even}(x)$$

  (where $\mathrm{Even}(x) :\equiv \exists y\colon x = y+y$). We would then have

  - $T \vdash \varphi(0)$; and
  - $T \nvdash \forall x\colon \varphi(x) \longrightarrow \varphi(x+1)$.
- But neither this matches mathematical practice.
  - When proving some universal fact

    'for all natural numbers n, …'

    by induction, it is usually (almost always?) the case that we can already prove each of its instances.
  - But here we have:

    $T \nvdash \varphi(n)$ for all odd natural numbers n.

This paragraph motivates the following definition and question.

¶ 5.5 DEFINITION

- Let T be an $\mathcal{L}^{\mathbf{OR}}$-theory.

- An $\mathcal{L}^{\mathbf{OR}}$-formula $\varphi(x)$ is *T-ω-provable* if and only if $T \vdash \varphi(n)$ for all natural numbers n.

¶ 5.6 FACT For all theories T of arithmetic: all T-inductive formulas are T-ω-provable.

¶ 5.7 QUESTION [Is the "ω-provability version" of Fact 5.2 still a fact?]

- Let T be a sound extension of PA^- such that $T \nvdash \mathrm{I}\mathcal{L}^{\mathbf{OR}}$.

- Is there then a T-ω-provable and non-trivially not T-inductive $\varphi(x)$ for which there is no T-inductive solution $\psi(x)$ such that

$$T \vdash \forall x\colon \psi(x) \longrightarrow \varphi(x)?$$

¶ 5.8 Here is a curious result, from joint work with my colleague Eric Johannesson.

¶ 5.9 FACT [Lundstedt and Johannesson (2021)] There are $\mathcal{L}^{\mathbf{OR}}$-formulas $\psi(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ such that:

- $\varphi_1(x)$ and $\varphi_2(x)$ are both PA^--ω-provable.

- $\varphi_1(x)$ and $\varphi_2(x)$ are both not PA^--inductive.

- $\psi(x)$ is a PA^--inductive solution for both $\varphi_1(x)$ and $\varphi_2(x)$.

- $\varphi_1(x)$ and $\psi(x)$ are *PA^--incomparable*:

$$PA^- \nvdash \forall x\colon \varphi_1(x) \longrightarrow \psi(x),$$

$$PA^- \nvdash \forall x\colon \psi(x) \longrightarrow \varphi_1(x).$$

- $\varphi_2(x)$ is *logically stronger and PA^--stronger* than $\psi(x)$:

$$\vdash \forall x\colon \varphi_2(x) \longrightarrow \psi(x),$$

$$PA^- \nvdash \forall x\colon \psi(x) \longrightarrow \varphi_2(x).$$

¶ 5.10 We prove Fact 5.9 by instantiating the preconditions of the following lemma.

¶ 5.11 LEMMA [Lundstedt and Johannesson (2021)]

- Let T be a sound extension of PA^- and let $\psi(x)$ be T-inductive.
- Suppose there is a model $M \vDash T$ and a c in M such that

  $$M \vDash \neg\psi(c) \wedge \psi(c{+}1).$$

- Then there are $\varphi_1(x)$ and $\varphi_2(x)$ such that:
  - $\varphi_1(x)$ and $\varphi_2(x)$ are both T-ω-provable and not T-inductive.
  - $\psi(x)$ is a T-inductive solution for both $\varphi_1(x)$ and $\varphi_2(x)$.
  - $\varphi_1(x)$ and $\psi(x)$ are *T-incomparable*:

    $$T \nvdash \forall x\colon \varphi(x) \longrightarrow \psi(x),$$

    $$T \nvdash \forall x\colon \psi(x) \longrightarrow \varphi(x).$$

  - $\varphi_1(x)$ is *logically stronger and T-stronger* than $\psi(x)$:

    $$\vdash \forall x\colon \varphi(x) \longrightarrow \psi(x),$$

    $$T \nvdash \forall x\colon \psi(x) \longrightarrow \varphi(x).$$

¶ 5.12 PROOF

- Since $\psi(x)$ is T-inductive, M and c are non-standard, and we have

  $$M \vDash \psi(c{+}1),\ \psi(c{+}2),\ \psi(c{+}3),\ \dots$$

  and

  $$M \vDash \neg\psi(c),\ \neg\psi(c{-}1),\ \neg\psi(c{-}2),\ \dots$$

- Define

  $$\varphi_1(x) :\equiv \psi(x \dot{-} 1) \leftrightarrow \psi(x \dot{-} 2)$$

  and

  $$\varphi_2(x) :\equiv \psi(x) \wedge \varphi_1(x),$$

  where '$\dot{-}$' denotes cutoff subtraction. (Cutoff subtraction is of course definable in any sound extension of PA^-).

- Then $\psi(x)$ is trivially a T-inductive solution to both $\varphi_1(x)$ and $\varphi_2(x)$.

- Also, since $\psi(x)$ is T-inductive and thus T-ω-provable, $\varphi_1(x)$ and $\varphi_2(x)$ are both T-ω-provable as well.

- We have:

	$x = c-2$	$x = c-1$	$x = c$	$x = c+1$	$x = c+2$
$M \models \psi(x)$	F	F	F	T	T
$M \models \psi(x \dot- 1)$			F	F	T
$M \models \psi(x \dot- 2)$			F	F	F
$M \models \varphi_1(x)$			T	T	F
$M \models \varphi_2(x)$				T	F

.

Thus:

- $\varphi_1(x)$ and $\varphi_2(x)$ are both not T-inductive since

$$M \not\models \varphi_1(c+1) \longrightarrow \varphi_1(c+2)$$

and

$$M \not\models \varphi_2(c+1) \longrightarrow \varphi_2(c+2).$$

- $\varphi_1(x)$ and $\psi(x)$ are T-incomparable since

$$M \not\models \varphi_1(c) \longrightarrow \psi(c)$$

and

$$M \not\models \psi(c+2) \longrightarrow \varphi_1(c+2).$$

- $\varphi_2(x)$ is logically stronger and T-stronger than $\psi(x)$ since

$$M \not\models \psi(c+2) \longrightarrow \varphi_2(c+2),$$

and, trivially,

$$\vdash \forall x\colon \varphi_2(x) \longrightarrow \psi(x).$$

□

¶ 5.13 I have no general recipe for coming up with T, $\psi(x)$, M and c satisfying the preconditions of Lemma 5.11, but the preconditions are satisfiable. My colleague Eric has come up with the following example.

¶ 5.14

- The *ordered ring of integer polynomials in one variable*, notation ‘$\mathbb{Z}[X]$’, consists of *formal polynomials*

  $$a_0+a_1X+a_2X^2+\cdots+a_nX^n$$

  where $n \geq 0$, a_0, …, a_n are integers, and $a_n \neq 0$ if $n > 0$.

- We define and denote addition, subtraction and multiplication as usual.

- ‘<’ denotes the order (‘>’ denotes its inverse), which is given by considering X as “infinitely large”——more precisely:

  - $a_0+a_1X+a_2X^2+\cdots+a_nX^n > 0$ if and only if $a_n > 0$; and

  - for any two p and q in $\mathbb{Z}[X]$: $p > q$ if and only if $p-q > 0$.

- The substructure of $\mathbb{Z}[X]$ consisting of the non-negative polynomials, notation ‘$\mathbb{Z}[X]^+$’, is a model of PA^-——that is, there is a substructure $\mathbb{Z}[X]^+ \vDash \mathrm{PA}^-$ of $\mathbb{Z}[X]$ with domain

  $$\{p \in \mathbb{Z}[X] : p \geq 0\}.$$

- A polynomial $a_0+a_1X+a_2X^2+\cdots+a_nX^n$ in $\mathbb{Z}[X]$ is *constant* if and only if $n = 0$.

- The standard numbers of $\mathbb{Z}[X]^+$ are the constant polynomials——that is, the natural numbers.

- The $\mathcal{L}^{\mathrm{OR}}$-formula

  $$\text{std-in-}\mathbb{Z}[X]^+ :\equiv \forall y{\leq}x\text{: Even}(y) \vee \text{Odd}(y)$$

  (where $\text{Even}(x) :\equiv \exists y\text{: } x = y+y$ and $\text{Odd}(x) :\equiv \exists y\text{: } x = y+y+1$) defines the set of standard numbers of $\mathbb{Z}[X]^+$ (since for any non-standard p in $\mathbb{Z}[X]^+$, there is an integer a such that $X+a < p$, and $\mathbb{Z}[X]^+ \nvDash \text{Even}(X+a) \vee \text{Odd}(X+a)$ for all integers a).

¶ 5.15 LEMMA [Lundstedt and Johannesson (2021)]

– Let

$$\psi(x) :\equiv \exists y \exists z\colon \text{std-in-}\mathbb{Z}[X]^{+}(z) \wedge x = y \times y + z.$$

– Then:

(a) $\psi(x)$ is PA^--inductive.

(b) $\mathbb{Z}[X]^{+} \vDash \neg\psi(X^2-1) \wedge \psi(X^2)$.

¶ 5.16 PROOF

(a) Quite trivial. Under PA^-, $\forall x.\psi(x)$ is basically an obfuscated version of

$$\forall x\colon \text{Even}(x) \vee \text{Odd}(x).$$

One may prove that $\psi(x)$ is PA^--inductive similarly to how one could easily prove that

$$\text{Even}(x) \vee \text{Odd}(x)$$

is. The details are omitted.

(b) – For any p in $\mathbb{Z}[X]^{+}$ we have

$$\mathbb{Z}[X]^{+} \vDash \psi(p)$$

if and only if

$$p = q^2+n$$

for some q and some natural number n.

– Thus clearly

$$\mathbb{Z}[X]^{+} \vDash \psi(X^2).$$

– Perhaps less clearly, but with details omitted nonetheless, we have

$$\mathbb{Z}[X]^{+} \vDash \neg\psi(X^2-1).$$ □

¶ 5.17 PROOF [of Fact 5.9] In Lemma 5.11, let $T = PA^-$, let $\psi(x)$ be as in Lemma 5.15, let $M = \mathbb{Z}[X]^{+}$, and let $c = X^2-1$. Then apply Lemma 5.15. □

¶ 5.18 An *analytic* proof is one that satisfies the subformula property. Analyticity and non-analyticity is important, I have come to understand, in inductive theorem proving. Thus we have the following variation of inductiveness.

¶ 5.19 DEFINITION

- Let T be an $\mathcal{L}^{\mathbf{OR}}$-theory and let $\varphi(x)$ be an $\mathcal{L}^{\mathbf{OR}}$-formula.

- $\varphi(x)$ is *analytically T-inductive* if and only if $\varphi(x)$ has a T-inductive solution that is a literal subformula of $\varphi(x)$ or an instance of a quantified subformula of $\varphi(x)$.

- If $\varphi(x)$ is not analytically T-inductive then $\varphi(x)$ is *non-trivially* not analytically T-inductive if and only if

$$T+\mathrm{I}\mathcal{L}^{\mathbf{OR}} \vdash \forall x.\varphi(x).$$

¶ 5.20 'Analytically T-inductive' is a wider notion than 'T-inductive'. For example, for T, $\varphi(x)$ and $\psi(x)$ from Fact 2.3, (2) showed that $\varphi(x)$ was not T-inductive. But since $\psi(x)$ is an instance of a quantified subformula of $\varphi(x)$, (3)–(5) shows that $\varphi(x)$ is analytically T-inductive.

¶ 5.21 Hetzl and Wong (2018) show as follows that there are non-trivially not analytically inductive formulas.

- Peano arithmetic proves the consistency of each of its $\mathrm{I}\Sigma_k$-fragments.

- We can take the consistency statements $\mathrm{Con}(\mathrm{I}\Sigma_k)$ as Π_1-sentences.

- For all natural numbers k:

$$\mathrm{PA} \vdash \mathrm{Con}(\mathrm{I}\Sigma_k) \text{ but } \mathrm{I}\Sigma_k \nvdash \mathrm{Con}(\mathrm{I}\Sigma_k).$$

- Hetzl and Wong on this result:

> Note that this result embodies a very strong non-analyticity requirement: given any $k \geq 1$, in order to prove $\mathrm{Con}(\mathrm{I}\Sigma_k)$ not only do we need a non-analytic induction formula, but we need one with more than k quantifier alternations even though $\mathrm{Con}(\mathrm{I}\Sigma_k)$ is only a Π_1 sentence.
> [Hetzl and Wong (2018), p. 5]

¶ 5.22 – Consider T, $\psi(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ from Proof 5.12.

– $\psi(x)$ is a T-inductive solution for both $\varphi_1(x)$ and $\varphi_2(x)$.

– $\psi(x)$ is a literal subformula of $\varphi_2(x)$. Thus $\varphi_2(x)$ is analytically T-inductive.

– Similarly, since $\psi(x \dot{-} 1)$ is a literal subformula of $\varphi_1(x)$, and since $\psi(x \dot{-} 1)$ is trivially a T-inductive solution for $\varphi_1(x)$ if $\psi(\mathrm{x})$ is, $\varphi_1(x)$ too is analytically inductive.

– Thus we have the following question.

¶ 5.23 QUESTION Does Fact 5.9 remain a fact if we replace ‘inductive’ with ‘analytically inductive’?

¶ 5.24 Of course, similar questions can be asked for many results concerning (non-)inductiveness.

References

Hetzl, Stefan, and Tin Lok Wong (2018): “Some observations on the logical foundations of inductive theorem proving”, Logical Methods in Computer Science 13(4), pp. 1–26 (corrected version of paper originally published Nov 16, 2017).

Lundstedt, Anders (2021): “Unprovability by straightforward induction”, in preparation.

Lundstedt, Anders, and Eric Johannesson (2021): “Non-straightforward induction proofs and the comparative strength of inductive solutions”, in preparation.