A NOTE ON NECESSARILY NON-ANALYTIC INDUCTION PROOFS

Anders Lundstedt (anders.lundstedt@philosophy.su.se)

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§0 Preliminaries

DEFINITIONS.

- L_0 is the first-order language with signature $\langle 0,1,+\rangle$.
- Any first-order language that is an expansion of L_0 is a *(first-order) language of arithmetic*.
- L_1 is the *(first-order) language of ordered rings*—signature $\langle 0,1,+,\cdot,<\rangle$.
- The *standard model of arithmetic* is the L_1-model with domain ℕ and the expected interpretation of each symbol.
- An L_1-model is *non-standard* if and only if it is not isomorphic to the standard model.
- Let L be a language of arithmetic and let M be an L-model.
  - The *standard natural numbers* of M are the denotations of all terms of the form

    $0+1+\cdots+1$ (0 or more 1's).
  - An element of M is a *non-standard natural number* if and only if it is not a standard natural number.
- The L_1-theory *true L_1-arithmetic*, notation T_1, is the theory of the standard model of arithmetic.
- Let L be a language of arithmetic and let $\varphi(x)$ be an L-formula in the free variable x. The *induction instance* for $\varphi(x)$ is the L-sentence

  $$\mathrm{IND}(\varphi) :\equiv \varphi(0)\land\forall x(\varphi(x)\to\varphi(x+1))\to\forall x.\varphi(x).$$
- Let L be a language of arithmetic and let T be an L-theory. Let $\varphi(x)$ be an L-formula in the free variable x. *T proves $\forall x.\varphi(x)$ by necessarily non-analytic induction* if and only if there is an L-formula $\psi(x)$ in the free variable x such that

  (1) $T,\mathrm{IND}(\varphi) \nvdash \forall x.\varphi(x)$,

  (2) $T \vdash \varphi(0)$,

  (3) $T \vdash \psi(0)$,

  (4) $T \vdash \forall x\colon \psi(x)\to\psi(x+1)$,

  (5) $T \vdash \forall x.\psi(x)\to\forall x.\varphi(x)$.

Under conditions (1)-(5) we also say that *ψ(x) witnesses that T proves ∀x.φ(x) by necessarily non-analytic induction*.

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§1 Introduction

The purpose of this note is to show that the following two facts must be proved by non-analytic induction in a rather strong sense.

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FACT 1.1. Define f : ℕ → ℕ recursively by

f(0) := 0,

f(n+1) := f(n)+2n+1.

Then f(n) is a perfect square for all natural numbers n.

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FACT 1.2. Define f : ℕ → ℕ recursively by

f(0) := 1,

f(n+1) := (n+1)f(n+1).

Define g : ℕ×ℕ → ℕ recursively by

g(0,m) := m,

g(n+1,m) := g(n,(n+1)m).

Then f(n) = g(n,1) for all natural numbers n.

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Even if we assume all of T_1 (that is, true arithmetic in the language ⟨0,1,+,·,<⟩), both these facts--if formulated by suitably expanding L_1 with function symbols--must be proved by non-analytic induction. This might seem counterintuitive: The involved functions are definable in T_1 and formulated using defining formulas, both facts are indeed theorems of T_1. But if we expand the language and add the defining equations for each function, then T_1 cannot prove that equations using the new function symbols are equivalent to their corresponding formulations using their defining formulas, since T_1 does not contain any induction axioms containing the new function symbols.

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The results can in fact be strengthened a bit further: Take any language L extending L_1 and consider the L-theory T of true arithmetic (suitably defined for that language). Then as long as we formulate the facts using fresh (that is, not in L) function symbols then T proves both facts by necessarily non-analytic induction. To avoid having to deal with how to define "true arithmetic" for arbitrary languages I will not prove these strengthenings.

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§2 The sum of any initial segment of the odd natural numbers is a perfect square

DEFINITIONS.

- The first-order language L_2 is L_1 expanded with the unary function symbol 'f_2'.

- $\varphi_2(x) :\equiv \exists y: f_2(x) = y^2$.

- $\psi_2(x) :\equiv f_2(x) = x^2$.

- The first-order theory T_2 is the L_2-theory consisting of T_1 together with the *the defining equations for f_2*

  $f_2(0) = 0$,

  $\forall x: f_2(x+1) = f_2(x)+2\cdot x+1$.

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FACT 2.1. $\psi_2(x)$ witnesses that T_2 proves $\forall x.\varphi_2(x)$ by necessarily non-analytic induction.

PROOF. We need to show (1)-(5) of the definition. (2)-(5) are easy. To show (1), that is

$$T_2,\mathrm{IND}(\varphi_2) \not\models \forall x.\varphi_2(x),$$

we will exhibit an L_2-model $M \models T_2$ with an element a such that $M \models \varphi_2(a)$ and $M \not\models \varphi_2(a+1)$. The following construction is due to Matt Kaufmann.

Let $M_0 \models T_1$ be a non-standard model and let a be any non-standard natural number in M_0. We expand M_0 to an L_2-model M by defining the interpretation $f_2^M : M_0 \to M_0$ as follows.

$$f_2^M(n) := n^2-9a^2 \quad \text{if } n = 5a+z \text{ for some integer } z,$$

$$f_2^M(n) := n^2 \qquad \text{otherwise.}$$

We need to verify that this definition is well-formed; that is, that n^2-9a^2 is never negative when $n = 5a+z$ for some integer z. We have

$$\begin{aligned} f_2^M(5a+z) &= (5a+z)^2-9a^2 \\ &= 25a^2+10za+z^2-9a^2 \\ &= 16a^2+10za+z^2 \\ &= a(16a+10z)+z^2 \end{aligned}$$

which is positive since

$a > 0$ (since a is non-standard),

$16a+10z > 0$ (since a is non-standard and z is not),

$z^2 \geq 0$ (since z^2 is a square).

With f_2^M so defined it is easy to verify that $M \vDash T_2$. We have $M \vDash \varphi_2(5a)$ since

$$\begin{aligned}\varphi_2(5a) &= (5a)^2-16a^2\\ &= 25a^2-16a^2\\ &= 9a^2\\ &= (3a)^2.\end{aligned}$$

It remains to show $M \nvDash \varphi_2(5a+1)$. We have

$$\begin{aligned}\varphi_2(5a+1) &= (5a+1)^2-16a^2\\ &= 25a^2+10a+1-16a^2\\ &= 9a^2+10a+1.\end{aligned}$$

This is not a perfect square in M since

$$\forall x \forall y: x > 0 \to 9x^2+10x+1 \neq y^2$$

is a theorem of T_1. Thus we have $M \nvDash \varphi_2(5a+1)$. □

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§3 A tail-recursive factorial is equivalent to the standard factorial

DEFINITIONS.

- The first-order language L_3 is L_1 expanded with the unary function symbols 'f_3' and 'g_3'.

- $\varphi_3(x) :\equiv f_3(x) = g_3(x,1)$.

- $\psi_3(x) :\equiv \forall y: y \cdot f_3(x) = g_3(x,y)$.

- The first-order theory T_3 is the L_3-theory consisting of T_1 together with the *the defining equations for f_3*

  $$f_3(0) = 1,$$

  $$\forall x: f_3(x+1) = (x+1) \cdot f_3(x)$$

  and *the defining equations for g_3*

  $$\forall y: g_3(0,y) = y,$$

  $$\forall x \forall y: g_3(x+1,y) = g_3(x,(x+1) \cdot y).$$

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FACT 3.1. $\psi_3(x)$ witnesses that T_3 proves $\forall x.\varphi_3(x)$ by necessarily non-analytic induction.

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To prove Fact 3.1 we will proceed as in the previous section. Conditions (2)-(5) of the definition are straightforward. To prove (1) we take any non-standard model $M_0 \vDash T_1$ and expand it to an L_3-model M, interpreting f_3 and g_3 such that we get $M \vDash T_3$ and a non-standard a such that $M \vDash \varphi_3(a)$ and $M \nvDash \varphi_3(a+1)$.

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ASSUMPTIONS.

- Let $M_0 \models T_1$ be a non-standard model.

- Let a be a non-standard number in M_0.

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We will expand M_0 to an L_3-model $M \models T_3$ such that $M \models \varphi_3(a-1)$ and $M \not\models \varphi_3(a)$. The defining equations for f_3 fix its interpretation on the standard natural numbers. For non-standard numbers c we will simply put $f_3^M(c) := 0$ (which does satisfy the defining equations). The defining equations for g_3 fix all its interpretations $g_3^M(n,m)$ where n is standard. For non-standard n we have quite some leeway. To achieve $M \models \varphi_3(a-1)$ and $M \not\models \varphi_3(a)$ we will put

$$g_3^M(a-1,1) := 0$$

and

$$g_3^M(a,1) := 1.$$

These together with the second defining equation for g_3 gives

$$\begin{aligned}0 &= g_3^M(a-1,1)\\ &= g_3^M(a-2,a-1)\\ &= g_3^M(a-3,(a-1)(a-2))\\ &= g_3^M(a-4,(a-1)(a-2)(a-3))\\ &\vdots\end{aligned}$$

respectively

$$\begin{aligned}1 &= g_3^M(a,1)\\ &= g_3^M(a-1,a)\\ &= g_3^M(a-2,a(a-1)),\\ &= g_3^M(a-3,a(a-1)(a-2))\\ &\vdots\end{aligned}$$

The defining equations impose no further restrictions so we can put $g_3^M(c,d) := 0$ for all yet to be defined cases.

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I will now make the construction in the preceding paragraph more precise.

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DEFINITION. $f_3^M : M_0 \to M_0$ is defined by

$$f_3^M(0) := 1,$$

$$f_3^M(n+1) := (n+1)f_3^M(n) \quad \text{if n is standard,}$$

$$f_3(c) := 0 \quad \text{if c is non-standard.}$$

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DEFINITIONS.

- The set $A \subseteq M_0 \times M_0$ is defined by

$$A := \{(n,m) : n,m \in M \text{ and } n \text{ is standard}\},$$

- The set $B \subseteq M_0 \times M_0$ is defined inductively by

$$\frac{}{(a,1) \text{ is in } B} \qquad \frac{(n+1,m) \text{ is in } B}{(n,(n+1)m) \text{ is in } B}\ .$$

- The set $C \subseteq M_0 \times M_0$ is defined by

$$C := M_0 \times M_0 - (A \cup B).$$

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REMARK. The elements of B are thus

(a,1),

(a-1,a),

(a-2,a(a-1)),

(a-3,a(a-1)(a-2)),

⋮

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LEMMA 3.2. A, B and C partitions $M_0 \times M_0$.

PROOF. Clearly:

- $A \cup B \cup C = M_0 \times M_0$.

- C is disjoint from A and from B.

It thus remains to show that A and B are disjoint. If (n,m) is in A then n is standard but if (n,m) is in B then n is non-standard. Thus A and B have no common elements and thus they are disjoint. □

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By Lemma 3.2 the following definition is well-formed.

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DEFINITION. $g_3^M : M_0 \times M_0 \to M_0$ is defined by

$$g_3^M(0,c) := c,$$

$$g_3^M(n+1,c) := g_3^M(n,(n+1)c) \quad \text{if } n \text{ is standard},$$

$$g_3^M(b,c) := 1 \quad \text{if } (b,c) \text{ is in } B,$$

$$g_3^M(b,c) := 0 \quad \text{if } (b,c) \text{ is in } C.$$

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LEMMA 3.3. Let X be A, B or C. For all n and m in X, (n+1,m) is in X if and only if (n,(n+1)m) is.

PROOF. It suffices to show this for X = A and X = B (since then it must hold for C, by definition of C).

- Case X = A:

  (n+1,m) is in A iff n+1 is standard
  iff n is standard
  iff (n,(n+1)m) is in A.

- Case X = B: The 'only if' direction holds by definition. For the 'if' direction, suppose (n,(n+1)m) is in B. Case splitting on the inductive definition of B gives the following.

  - Case (n,(n+1)m) = (a,1): Then n = a and (n+1)m = 1, so (a+1)m = 1 so a+1 = m = 1, so a = 0. Since a is non-standard this is a contradiction so this case is not possible.

  - Case (n,(n+1)m) = (n',(n'+1)m') for some n' and m' in M_0 such that (n'+1,m') is in B: Trivial. □

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LEMMA 3.4.

(1) $M \models T_3$.

(2) $M \models \varphi_3(a-1)$.

(3) $M \not\models \varphi_3(a)$.

PROOF.

(1) We need to show:

  (a) $M \models T_1$,

  (b) $M \models f_3(0) = 1$,

  (c) $M \models \forall x$: $f_3(x+1) = (x+1)\cdot f_3(x)$

  (d) $M \models \forall y$: $g_3(0,y) = y$,

  (e) $M \models \forall x\forall y$: $g_3(x+1,y) = g_3(x,(x+1)\cdot y)$.

  (a) holds since M is an M_0-expansion. (b) and (d) holds by definition of f_3^M and g_3^M, respectively. It remains to show (c) and (e).

  (c) Suppose n is in M. We want to show

  $$f_3^M(n+1) = (n+1)f_3^M(n).$$

  If n is standard then this holds by definition. If n is non-standard then, by definition, both sides of the equation are 0 and thus equal.

(e) Suppose n and m is in M_0. We need to show

(*) $g_3^M(n+1,m) = g_3^M(n,(n+1)m)$.

By Lemma 3.2 it suffices to consider the following cases.

- Case (n+1,m) is in A: (*) holds by definition of g_3^M.

- Case (n+1,m) is in B: Then, by Lemma 3.3, (n,(n+1)m) is in B as well so, by definition of g_3^M, both sides of (*) are 1 and thus (*) holds.

- Case (n+1,m) is in C: Similar to the previous case.

(2) We need to show

(*) $f_3^M(a-1) = g_3^M(a-1,1)$.

We have $f_3^M(a-1) = 0$ by definition. We have that (a-1,1) is in C, so $g_3^M(a-1) = 0$ by definition. Thus (*) holds.

(3) We need to show

(*) $f_3^M(a) \neq g_3^M(a,1)$.

Again, we have $f_3^M(a) = 0$ by definition. We have that (a,1) is in B, so $g_3^M(a,1) = 1$ by definition. Thus (*) holds. □

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PROOF OF FACT 3.1. We need to show (1)-(5) of the definition. (1) follows from Lemma 3.4. (2)-(5) are easy to verify. □

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§4 Conclusion

These results are improvements of previous results, where in Facts 2.1 and 3.1 we had PA^- instead of T_1 (PA^- is "the theory of the non-negative parts of nontrivial discretely ordered commutative rings"). Naturally, one might wonder if it is always the case that such results can be strengthened by replacing PA^- with T_1. This is not the case for trivial reasons: If we formulate Facts 1.1 and 1.2 in L_1--that is, using defining formulas and not fresh function symbols--then both formulations are of course theorems of T_1. Requiring use of an expanded language would of course not suffice either, since we could just add some validity in the expanded language to our formulas. But perhaps there is some criterion for when it is indeed the case that PA^- can be replaced with T_1. That would be a really interesting result and is something I think is worth looking into.