REFLECTIONS ON THE TERMINOLOGY OF "NECESSARILY NON-ANALYTIC INDUCTION PROOFS"

Anders Lundstedt (anders.lundstedt@philosophy.su.se)

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In my research project with the preliminary title "Necessarily non-analytic induction proofs" I investigate the phenomenon where some mathematical facts seem to not lend themselves to a "straightforward" induction proof. Sometimes it does not seem possible to prove a fact $\forall x.\varphi(x)$ by induction with $\varphi(x)$ as induction hypothesis. Instead what works is to prove some other fact $\forall x.\psi(x)$ by induction (with $\psi(x)$ as induction hypothesis), and $\forall x.\varphi(x)$ then follows from $\forall x.\psi(x)$. Typically this proof method is called something like "strengthening of the induction hypothesis". However, there need not always be any precise sense in which $\psi(x)$ is stronger than $\varphi(x)$. Thus a more general terminology is wanted.

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Hetzl and Wong (2017) have made precise sense of what it would mean that a fact cannot be proved by "straightforward induction". They proved that "inductive theorem proving requires non-analytic induction axioms", which can be phrased precisely as follows. An induction axiom

$$\varphi(0)\wedge\forall x(\varphi(x)\to\varphi(x+1))\to\forall x.\varphi(x)$$

is *non-analytic* for a sentence σ if $\varphi(x)$ is not an instance of a subformula of σ. Then there are consequences $PA \vdash \sigma$ any derivation of which must make use of induction axioms that are non-analytic for σ.

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Based on the notions introduced by Hetzl and Wong, I have made the following definitions (Lundstedt, 2020).

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DEFINITIONS.

- A first-order language is a *(first-order) language of arithmetic* if and only if it is an expansion of the language $\langle 0,1,+\rangle$.

- Let L be a language of arithmetic.

  - Let $\varphi(x)$ be an L-formula in the free variable x. The *induction instance* for φ is the L-sentence

$$IND(\varphi) :\equiv \varphi(0)\wedge\forall x(\varphi(x)\to\varphi(x+1))\to\forall x.\varphi(x).$$

- Let T be an L-theory and let $\varphi(x)$ and $\psi(x)$ be L-formulas in the free variable x. Say that *$\psi(x)$ witnesses that T proves $\forall x.\varphi(x)$ by necessarily non-analytic induction* if and only if

  (1) $T, IND(\varphi) \nvdash \forall x.\varphi(x)$,

  (2) $T \vdash \varphi(0)$,

  (3) $T \vdash \psi(0)$,

  (4) $T \vdash \forall x: \psi(x) \rightarrow \psi(x+1)$,

  (5) $T \vdash \forall x.\psi(x) \rightarrow \forall x.\varphi(x)$.

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As used here, there are at least two problems with the terminology "T proves $\forall x.\varphi(x)$ by necessarily non-analytic induction":

(P1) If (1)-(5) holds then T does not actually prove $\forall x.\varphi(x)$——rather it is T together with an appropriate induction axiom that prove $\forall x.\varphi(x)$. Thus "T proves $\forall x.\varphi(x)$ ..." is a bit misleading.

(P2) It is possible to construct T, $\varphi(x)$ and $\psi(x)$ such that (1)-(5) holds while $IND(\psi)$ is not non-analytic for $\forall x.\varphi(x)$. Indeed, suppose we have:

> $\psi(x)$ witnesses that T proves $\forall x.\varphi(x)$ by necessarily non-analytic induction.

Suppose we also have the following strengthening of (5):

(5') $T \vdash \forall x: \psi(x) \rightarrow \varphi(x)$.

(See for example my summary of results (Lundstedt, 2020) for cases where (5') holds in addition to (1)-(5).) Then it is an easy exercise to verify that we have:

> $\psi(x)$ witnesses that T proves $\forall x: \varphi(x) \vee \psi(x)$ by necessarily non-analytic induction.

But $\psi(x)$ is a subformula of $\varphi(x) \vee \psi(x)$ and thus $IND(\psi)$ is not non-analytic for $\forall x: \varphi(x) \vee \psi(x)$. Thus we should not say that T proves $\forall x: \varphi(x) \vee \psi(x)$ by necessarily non-analytic induction.

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I think a good solution to (P2) would be to simply replace 'non-analytic' with 'non-straightforward'. For solving (P1) I could change the terminology to something like "T IND-proves $\forall x.\varphi(x)$ ..." where IND-proves would mean that T proves $\forall x.\varphi(x)$ in a first-order logic extended with an induction rule. Right now

I do not know whether this would be a good solution. To avoid extra work I will keep the terminology as it is until I have solutions to both (P1) and (P2) that I am happy with.

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I will probably replace 'non-analytic' with 'non-straightforward'. But perhaps it would also make sense to keep an alternative version of the definition where we keep 'non-analytic' and replace (1) with

(1') $T,IND(\beta) \nvdash \forall x.\varphi(x)$ for all instances $\beta(x)$ of subformulas of $\varphi(x)$.

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OPEN QUESTION: Would there be any point in point in having such an alternative definition? Put differently, would there be any point in distinguishing between 'necessarily non-analytic' and the more general 'necessarily non-straightforward'?

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Another terminological problem concerns the title of the research project, even if changed to "Necessarily non-straightforward induction proofs". The problem is roughly that it is superfluous to call a non-straightforward induction proof 'necessarily non-straightforward'. On the most reasonable (at least in my opinion) literal reading, any non-straightforward induction proof is necessarily non-straightforward, since any straightforward induction proof of the same fact would be a different proof.

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I think the following radical change of terminology would solve all above problems. Simply call a consequence $T \vdash \forall x.\varphi(x)$ 'non-straightforward' if any derivation of it must make use of induction axioms other than $IND(\varphi)$. (We could define 'non-analytic' consequences similarly.) A drawback with this solution is that 'non-straightforward consequence' is quite non-descriptive.

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References

Hetzl, Stefan and Tin Lok Wong (2017): "Some observations on the logical foundations of inductive theorem proving", Logical Methods in Computer Science 13(4).

Lundstedt, Anders (2020): "Necessarily non-analytic induction proofs——summary of some results", manuscript, version 2020-01-22, https://anderslundstedt.com/research/non-analytic-induction/all-files.html.