
NECESSARILY NON-ANALYTIC INDUCTION--SUMMARY OF SOME RESULTS

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§1 Introduction

Sometimes when trying to prove a fact F by induction one gets "stuck" when trying to prove the induction step. A solution is sometimes to instead prove a "stronger" fact S by induction. This proof method is usually called something like "strengthening of the induction hypothesis". However, there need not always be a precise sense in which the fact S is "stronger". Thus, following Hetzl and Wong (2018), we use the more general terminology "non-analytic induction proofs" for such proofs. A natural question for such proofs is whether the non-analyticity is necessary--that is, whether one could prove F without a "detour" proving some other fact. Hetzl and Wong have made precise sense of this question for first-order theories and sentences of arithmetic.

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In §2 we give two examples of non-analytic induction proofs, proving the two facts:

(F1) For all natural numbers n : $1+3+5+\dots+(2n-1) = k^2$ for some natural number k .

(F2) For all natural numbers n : $1+1/2^2+1/3^2+\dots+1/(n+1)^2 < 2$.

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In §3 we present a slight generalization and reformulation of some of the notions introduced by Hetzl and Wong.

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In §4 we show, using the notions introduced in §3, that there is a precise sense in which (F1) must be proved by non-analytic induction.

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In §5 we present some preliminary results towards settling whether (F2) must be proved by non-analytic induction in any precise sense afforded by the notions introduced in §3.

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In §6 we present some ideas for future work.

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§2 Two results proved by non-analytic induction

For our first case, we have that the sum of any initial segment of the odd natural numbers is a perfect square, that is,

for all natural numbers n : $1+3+5+\dots+(2n-1) = k^2$ for some natural number k ,

or, avoiding ellipsis notation, the following.

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DEFINITION. $f_1 : \mathbb{N} \rightarrow \mathbb{N}$ is recursively defined by

$$f_1(0) := 0,$$

$$f_1(n+1) := f_1(n)+2n+1.$$

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FACT 1. For all natural numbers n : $f_1(n) = k^2$ for some natural number k .

PROOF ATTEMPT. "Straightforward induction":

- Base case: $f_1(0) = 0 = 0^2$ (by definitions) so $f_1(0) = k^2$ for $k = 0$.

- Induction step:

$$\begin{aligned} f_1(n+1) &= f_1(n)+2n+1 && \text{(by definition)} \\ &= k^2+2n+1 && \text{(for some } k, \text{ by induction hypothesis)}. \end{aligned}$$

But k^2+2n+1 is not a perfect square for all natural numbers k and n so how do we proceed from here?

ACTUAL PROOF. It suffices to prove the following "stronger" fact, of which Fact 1 is a logical consequence.

$$f_1(n) = n^2 \text{ for all natural numbers } n.$$

(This fact is not a logical consequence of Fact 1, so it is stronger than Fact 1 in at least that sense.)

We prove this fact by induction.

- Base case: $f_1(0) = 0^2 \equiv 0 = 0$ (by definitions).

- Induction step:

$$\begin{aligned} f_1(n+1) &= f_1(n)+2n+1 && \text{(by definition)} \\ &= n^2+2n+1 && \text{(by induction hypothesis)} \\ &= (n+1)^2 && \text{(by elementary arithmetic)}. \end{aligned}$$

□

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For our second case, we have

$$1+1/2^2+1/3^2+\dots+1/(n+1)^2 < 2 \text{ for all natural numbers } n,$$

or, avoiding ellipsis notation, the following.

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DEFINITION. $f_2 : \mathbb{N} \rightarrow \mathbb{Q}$ is recursively defined by

$$f_2(0) := 1,$$

$$f_2(n+1) := f_2(n) + 1/(n+2)^2.$$

**

FACT 2. $f_2(n) < 2$ for all natural numbers n .

PROOF ATTEMPT. "Straightforward" induction:

- Base case: $f_2(0) < 2 \equiv 1 < 2$ (by definition).

- Induction step:

$$\begin{aligned} f_2(n+1) &= f_2(n) + 1/(n+2)^2 && \text{(by definition)} \\ &< 2 + 1/(n+2)^2 && \text{(by induction hypothesis)}. \end{aligned}$$

But $2 + 1/(n+2)^2 \not< 2$ for any natural number n so how do we proceed from here?

ACTUAL PROOF. It clearly suffices to prove the "stronger" fact

$$f_2(n) \leq 2 - 1/(n+1) \text{ for all natural numbers } n.$$

(In what sense is this fact stronger than Fact 2? It is at least stronger in the sense that for all $f : \mathbb{N} \rightarrow \mathbb{Q}$,

$$f(n) \leq 2 - 1/(n+1) \text{ for all } n$$

implies

$$f(n) < 2 \text{ for all } n,$$

while the converse implication does not hold for all $f : \mathbb{N} \rightarrow \mathbb{Q}$.)

We prove this fact by induction.

- Base case: $f_2(0) \leq 2 - 1/(0+1) \equiv 1 \leq 1$ (by definitions).

- Induction step:

$$\begin{aligned} f_2(n+1) &= f_2(n) + 1/(n+2)^2 && \text{(by definition)} \\ &\leq 2 - 1/(n+1) + 1/(n+2)^2 && \text{(by induction hypothesis)} \\ &= 2 - 1/(n+2) - 1/(n+1)(n+2)^2 && \text{(by lots of elementary arithmetic)} \\ &\leq 2 - 1/(n+2) && \text{(by more or less obvious facts). } \square \end{aligned}$$

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§3 Definitions

What follows in this section is a reformulation and slight generalization of some of the notions introduced by Hetzl and Wong (2017).

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DEFINITIONS.

- The **full (first-order) language of arithmetic**, notation $\mathcal{L}[\text{full}]$, is the first-order language that for each natural number n has
 - a constant symbol n ,
 - a function symbol f of arity $n+1$ for each function $f : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$,
 - a relation symbol P of arity n for each relation $P \subseteq \mathbb{N}^n$.
- The **minimal (first-order) language of arithmetic**, notation $\mathcal{L}[\text{min}]$, is the $\mathcal{L}[\text{full}]$ -reduct with signature $\langle 0, 1, +, \cdot, < \rangle$.
- A first-order language L is a **(first-order) language of arithmetic** if and only if L is an $\mathcal{L}[\text{min}]$ -expansion and an $\mathcal{L}[\text{full}]$ -reduct.

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Thus a first-order language of arithmetic has as symbols natural numbers and operations and relations on natural numbers. This is just a convenient "trick" which allows us to easily define the standard model.

**

DEFINITIONS. Let L be a language of arithmetic.

- The **standard L-model** has domain \mathbb{N} and each symbol interpreted as itself.
- An L -model not isomorphic to the standard L -model is a **non-standard L-model**.
- The L -theory **true L-arithmetic** is the theory of the standard L -model.
- Any subset of true L -arithmetic is a **theory of L-arithmetic**.

**

DEFINITION. Let L be a language of arithmetic and let $\varphi(x)$ be an L -formula in the free variable x . The **induction instance** for $\varphi(x)$ is the L -sentence

$$\text{IND}(\varphi) := \varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x. \varphi(x).$$

**

DEFINITION. Let L be a language of arithmetic and let T be an L -theory. Let $\varphi(x)$ be an L -formula in the free variable x . Say that **T proves $\forall x. \varphi(x)$ by necessarily non-analytic induction** if and only if there is an L -formula $\psi(x)$ in the free variable x such that

- (1) $T, \text{IND}(\varphi) \not\vdash \forall x. \varphi(x)$,
- (2) $T \vdash \varphi(0)$,
- (3) $T \vdash \psi(0)$,
- (4) $T \vdash \forall x: \psi(x) \rightarrow \varphi(x+1)$,
- (5) $T \vdash \forall x. \psi(x) \rightarrow \forall x. \varphi(x)$.

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REMARK. The above definition does not require T to be a theory of arithmetic (that is, T could contain sentences false in the standard model). We will of course require T to be a theory of arithmetic when using this definition to make precise sense of the necessary non-analyticity of non-analytic induction proofs (of arithmetic) found in the wild. But we do not rule out that there might be interesting uses of the above definition with T not a theory of arithmetic.

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When trying to prove that typical T and $\varphi(x)$ satisfy the above definition, the hardest part will be--as far as we can tell at least--to show condition (1). Therefore it makes sense to break out a lemma that captures the idea of using non-standard models to establish (1).

**

LEMMA. Let L be a language of arithmetic. Let $\varphi(x)$ be an L -formula in the free variable x such that $\forall x.\varphi(x)$ is true in the standard L -model. Let T be an L -theory such that $T \vdash \varphi(0)$. The following are then equivalent.

- $T, \text{IND}(\varphi) \not\vdash \forall x.\varphi(x)$.
- There is a non-standard L -model $M \models T$ and a non-standard number c in M such that $M \models \varphi(c)$ and $M \not\models \varphi(c+1)$.

PROOF. By soundness and completeness,

$$T, \text{IND}(\varphi) \not\vdash \forall x.\varphi(x)$$

is equivalent to

$$\text{there is } M \models T, \text{IND}(\varphi) \text{ such that } M \not\models \forall x.\varphi(x)$$

which, since the conclusion of $\text{IND}(\varphi)$ then is false in M , is equivalent to

$$\text{there is } M \models T \text{ such that } M \not\models \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)) \text{ and } M \not\models \forall x.\varphi(x)$$

which, since $M \models T \vdash \varphi(0)$, is equivalent to

$$\text{there is } M \models T \text{ such that } M \not\models \forall x(\varphi(x) \rightarrow \varphi(x+1)) \text{ and } M \not\models \forall x.\varphi(x)$$

which is equivalent to

$$\text{there is } M \models T \text{ and } c \text{ in } M \text{ such that } M \not\models \varphi(c) \rightarrow \varphi(c+1) \text{ and } M \not\models \forall x.\varphi(x)$$

which is equivalent to

$$\text{there is } M \models T \text{ and } c \text{ in } M \text{ such that } M \models \varphi(c) \text{ and } M \not\models \varphi(c+1) \text{ and } M \not\models \forall x.\varphi(x)$$

which, dropping the now redundant $M \not\models \forall x.\varphi(x)$, is equivalent to

$$\text{there is } M \models T \text{ and } c \text{ in } M \text{ such that } M \models \varphi(c) \text{ and } M \not\models \varphi(c+1)$$

which, since $\forall x.\varphi(x)$ holds in the standard model, is equivalent to

$$\text{there is nonstandard } M \models T \text{ and a non-standard number } c \text{ in } M \text{ such that } M \models \varphi(c) \text{ and } M \not\models \varphi(c+1).$$

□

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DEFINITION. The $\mathcal{L}[\text{min}]$ -theory PA^- is axiomatized by the following.

- 0 and 1 are distinct:

$$0 \neq 1.$$

- Associativity of addition and multiplication:

$$(x+y)+z = x+(y+z),$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

- Commutativity of addition and multiplication:

$$x+y = y+x,$$

$$x \cdot y = y \cdot x.$$

- Distributivity of addition over multiplication:

$$x \cdot (y+z) = x \cdot y + x \cdot z.$$

- 0 is an additive identity and a multiplicative zero:

$$x \cdot 0 = 0.$$

- The order is irreflexive:

$$x \not< x.$$

- The order is transitive:

$$x < y \wedge y < z \rightarrow x < z.$$

- The order is total:

$$x < y \vee x = y \vee y < x.$$

- The order is discrete:

$$x = 0 \vee x = 1 \vee 1 < x.$$

- Addition and multiplication respect the order:

$$x < y \rightarrow x+z < y+z,$$

$$x < y \wedge 0 \neq z \rightarrow x \cdot z < y \cdot z.$$

- Smaller elements can be subtracted from larger:

$$x < y \rightarrow \exists z. x+z = y.$$

**

FACTS.

- For all $\mathcal{L}[\text{min}]$ -structures M : $M \models \text{PA}^-$ if and only if M is the non-negative part of a nontrivial discretely ordered commutative ring.

- PA^- is a theory of $\mathcal{L}[\min]$ -arithmetic.
- $PA \equiv PA^- \cup \{IND(\varphi) : \varphi \text{ an } \mathcal{L}[\min]\text{-formula in one free variable}\}$.

**

§4 Fact 1 must be proved by non-analytic induction

DEFINITION. $\mathbb{Z}[X] := \langle \mathbb{Z}[X], 0, 1, +, \cdot, < \rangle$ is the ordered ring of polynomials in the indeterminate X with coefficients in \mathbb{Z} .

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Elements of $\mathbb{Z}[X]$ are polynomials

$$z_0 + z_1X + z_2X^2 + \cdots + z_nX^n$$

with z_0, z_1, \dots, z_n in \mathbb{Z} and if $n \neq 0$ then $z_n \neq 0$. These can be divided into the **constants** polynomials

$$z \quad (z \text{ in } \mathbb{Z})$$

and the **non-constant** polynomials

$$z + pX \quad (p \text{ in } \mathbb{Z}[X], z \text{ in } \mathbb{Z}).$$

**

Addition, multiplication and subtraction in $\mathbb{Z}[X]$ are as expected. The order is given by

$$z_0 + z_1X + z_2X^2 + \cdots + z_nX^n > 0 \quad \text{if and only if} \quad z_n > 0,$$

$$p > q \quad \text{if and only if} \quad p - q > 0.$$

**

DEFINITION. $\mathbb{Z}[X]^+ = \langle \mathbb{Z}[X]^+, 0, 1, +, \cdot, < \rangle$ is the non-negative part of $\mathbb{Z}[X]$, that is, a polynomial p from $\mathbb{Z}[X]$ is in $\mathbb{Z}[X]^+$ if and only if $p \geq 0$.

**

Elements of $\mathbb{Z}[X]^+$ are polynomials

$$z_0 + z_1X + z_2X^2 + \cdots + z_nX^n$$

with z_0, z_1, \dots, z_n in \mathbb{Z} and $z_n \geq 0$ and if $n \neq 0$ then $z_n \neq 0$. The constant polynomials are

$$n \quad (n \text{ in } \mathbb{N})$$

and the non-constant polynomials are

$$z + pX \quad (p \text{ in } \mathbb{Z}[X], z \text{ in } \mathbb{Z}, p > 0).$$

**

FACT. $\mathbb{Z}[X]^+ \equiv PA^-$.

PROOF. $\mathbb{Z}[X]^+$ is the non-negative part of the nontrivial discretely ordered commutative ring $\mathbb{Z}[X]$. □

**

DEFINITION. The language of arithmetic L_1 is $\mathcal{L}[\min]$ expanded with the function symbol f_1 .

**

DEFINITION.

- The theory T_1 of L_1 -arithmetic is defined by

$$T_1 := \text{PA}^-, f_1(0) = 0, \forall x. f_1(x+1) = f_1(x) + 2x + 1.$$

- The L_1 -formulas $\varphi_1(x)$ and $\psi_1(x)$ are defined by

$$\varphi_1(x) := \exists y. f_1(x) = y \cdot y,$$

$$\psi_1(x) := f_1(x) = x \cdot x.$$

**

FACT. T_1 proves $\forall x. \varphi_1(x)$ by necessarily non-analytic induction.

PROOF. We verify conditions (1)-(5) with $\psi = \psi_1$.

(2) Trivial.

(3) Trivial.

(4) The proof of Fact 1 given earlier can be straightforwardly formalized in PA^- .

(5) Trivial.

(1) We appeal to the lemma in §3 and exhibit a non-standard L_1 -model $M \models T_1$ and an element c in M such that $M \models \varphi(c)$ and $M \not\models \varphi(c+1)$. Let M be the L_1 -expansion of $\mathbb{Z}[X]^+$ with f_1 interpreted as follows.

$$f_1^M(0) := 0,$$

$$f_1^M(n+1) := f_1^M(n) + 2n + 1,$$

$$f_1^M(pX-1) := pX^2$$

$$f_1^M(pX-1+(n+1)) := f_1^M(pX-1+n) + 2(pX-1+n) + 1,$$

$$f_1^M(pX-1-(n+1)) := f_1^M(pX-1-n) - 2(pX-1-n) + 1.$$

The right hand side of the last equation does indeed define a polynomial of $\mathbb{Z}[X]^+$ since by construction the degree of $f_1^M(z+pX)$ is greater than the degree of pX for all integers z and all p in $\mathbb{Z}[X]^+$. (We get the last equation by solving

$$f_1^M(pX-1-n) = f_1^M(pX-1-(n+1)+1) = f_1^M(pX-1-(n+1)) + 2(pX-1-(n+1)) + 1$$

for $f_1^M(pX-1-(n+1))$.)

We then have

$$f_1^M(X-1) = X^2$$

which is a perfect square in M and we have

$$\begin{aligned} f_1^M(X) &= f_1^M(X-1)+2(X-1)+1 \\ &= X^2+2X-1 \end{aligned}$$

which is not a perfect square in M . Thus we have $M \models \phi_1(X)$ and $M \not\models \phi_1(X+1)$. By construction we also have $M \models T_1$ so we are done. \square

**

§5 Must Fact 2 be proved by non-analytic induction?

Since Fact 2 is a statement involving rationals, a little work is needed to phrase it as a natural statement in first-order arithmetic (that is, without involving any coding).

**

DEFINITION. $g : \mathbb{N} \rightarrow \mathbb{N}$ and $h : \mathbb{N} \rightarrow \mathbb{N}$ are recursively defined by

$$\begin{aligned} g(0) &:= 1, \\ g(n+1) &:= (n+2)^2g(n)+h(n), \\ h(0) &:= 1, \\ h(n+1) &:= (n+2)^2h(n). \end{aligned}$$

**

We have

$$f_2(n) = g(n)/h(n)$$

so the inequality $f_2(n) < 2$ can be rewritten as

$$g(n) < 2h(n).$$

Similarly the inequality $f_2(n) \leq 2-1/(n+1)$ becomes

$$(n+1)g(n) \leq (2n+1)h(n).$$

**

All in all, we have the following rephrasings of Fact 2 and its proof attempt and proof.

**

LEMMA. For all natural numbers n , if $(n+1)g(n) \leq (2n+1)h(n)$ then $g(n) < 2h(n)$.

PROOF. Suppose $(n+1)g(n) \leq (2n+1)h(n)$. Since $(2n+1)h(n) < 2(n+1)h(n)$ we then have $(n+1)g(n) < 2(n+1)h(n)$ so we must have $g(n) < 2h(n)$. \square

**

FACT 2'. $g(n) < 2h(n)$ for all n .

PROOF ATTEMPT. Induction on n .

- Base case:

$$\begin{aligned} g(0) < 2h(0) &\equiv 1 < 2 \cdot 1 \quad (\text{by definition}) \\ &\equiv 1 < 2. \end{aligned}$$

- Induction step:

$$\begin{aligned} g(n+1) &= (n+2)^2 g(n) + h(n) \quad (\text{by definition}) \\ &< (n+2)^2 \cdot 2h(n) + h(n) \quad (\text{by induction hypothesis}) \end{aligned}$$

But for any n we have $(n+2)^2 \cdot 2h(n) + h(n) \not< 2h(n)$. Thus we are "stuck".

ACTUAL PROOF. By the lemma, it suffices to prove

$$(n+1)g(n) \leq (2n+1)h(n) \text{ for all } n.$$

We prove this by induction on n .

- Base case:

$$\begin{aligned} (0+1)g(0) &\leq (2 \cdot 0 + 1)h(0) \quad (\text{by definition}) \\ &\equiv 1 \cdot 1 \leq 1 \cdot 1 \\ &\equiv 1 \leq 1. \end{aligned}$$

- Induction step: Our induction hypothesis is

$$(IH) \quad (n+1)g(n) \leq (2n+1)h(n)$$

and we must show

$$(GOAL) \quad (n+2)g(n+1) \leq (2n+3)h(n+1).$$

We have

$$\begin{aligned} (n+2)g(n+1) &= (n+2)((n+2)^2 g(n) + h(n)) \\ &= (n+2)^3 g(n) + (n+2)h(n) \end{aligned}$$

and

$$\begin{aligned} (2n+3)h(n+1) &= (2n+3)(n+2)^2 h(n) \\ &= (2n+3)(n+2)(n+2)h(n) \\ &= (2n^2+7n+6)(n+2)h(n) \\ &= (2n^2+7n+5)(n+2)h(n) + (n+2)h(n) \end{aligned}$$

so (GOAL) is equivalent to

$$(n+2)^3 g(n) \leq (2n^2+7n+5)(n+2)h(n),$$

which is equivalent to

$$(n+2)^2 g(n) \leq (2n^2+7n+5)h(n),$$

which we have by

$$\begin{aligned}
 (n+2)^2 g(n) &= ((n+1)+1)^2 g(n) \\
 &= ((n+1)^2 + 2(n+1) + 1)g(n) \\
 &= ((n+1)^2 + 2(n+1))g(n) + g(n) \\
 &= (n+3)(n+1)g(n) + g(n) \\
 &\leq (n+3)(2n+1)h(n) + g(n) && \text{(by (IH))} \\
 &\leq (n+3)(2n+1)h(n) + 2h(n) && \text{(by Lemma and (IH))} \\
 &= (2n^2 + 7n + 5)h(n). && \square
 \end{aligned}$$

**

REMARK. We find it a bit curious that the proof we gave of Fact 2 uses only one application of the induction hypothesis, while the above proof of Fact 2' uses two applications. Is there a proof of Fact 2' which, in some precise sense, uses only one application of the induction hypothesis?

**

DEFINITION.

- The language of arithmetic L_2 is $\mathcal{L}[\min]$ expanded with the function symbols g and h .
- The L_2 -sentences $\text{DEF}(g)$ and $\text{DEF}(h)$ and the L_2 -formulas $\varphi_2(x)$ and $\psi_2(x)$ are defined by

$$\text{DEF}(g) := g(0) = 1 \wedge \forall x. g(x+1) = (x+2) \cdot (x+2) \cdot g(x) + h(x)$$

$$\text{DEF}(h) := h(0) = 1 \wedge \forall x. h(x+1) = (x+2) \cdot (x+2) \cdot h(x),$$

$$\varphi_2(x) := g(x) < 2 \cdot h(x),$$

$$\psi_2(x) := (x+1) \cdot g(x) + h(x) \leq 2 \cdot (x+1) \cdot h(x).$$

**

CONJECTURE. There is an L_2 -theory $T \supseteq \text{PA}^-$ of arithmetic such that

- (1) $T \vdash \text{DEF}(g)$,
- (2) $T \vdash \text{DEF}(h)$,
- (3) T proves $\forall x. \varphi_2(x)$ by necessarily non-analytic induction.

**

Adapting the proof of the previous section—cleverly interpreting function symbols on $\mathbb{Z}[X]^+$ —will not work to settle the above conjecture, as the unique L_2 -expansion M of $\mathbb{Z}[X]^+$ that satisfies $\text{DEF}(g)$ and $\text{DEF}(h)$ has $g^M(p) = h^M(p) = 0$ for all non-constant polynomials p . In fact, if $M \models \text{DEF}(h)$ is the non-negative part of a discretely ordered commutative polynomial ring $R[X]$ then $h^M(p) = 0$ for all non-constant polynomials—for if $h^M(p) \neq 0$ for some non-constant polynomial p then the degrees of $h^M(p)$, $h^M(p-1)$, $h^M(p-2)$, ... would form an infinitely descending chain of natural numbers. It does not help if M instead is the non-negative part of a discretely ordered commutative ring $R[X, X^{-1}]$ of Laurent polynomials—in that case if $h^M(p) \neq 0$ for some non-constant polynomial p then there is a natural number n such that the degree of $h^M(p-n)$ is less than its order. For more details see §A.

**

§6 Future work

One line of future work would of course be to settle the conjecture in §5.

**

In §4 we expanded $\mathbb{Z}[X]^+$ to an L_1 -model M in order to show that T_1 proves $\forall x.\phi_1(x)$ by necessarily non-analytic induction. For each sentence σ of true L_1 -arithmetic false in M it is natural to ask whether adding σ to T_1 lets us prove $\forall x.\phi_1(x)$ with an analytic induction proof. One simple sentence of true L_1 -arithmetic that is false in M is "all numbers are odd or even", that is

$$\sigma_1 := \forall x \exists y: x = y+y \vee x = y+y+1.$$

**

CONJECTURE. $T_1 \cup \{\sigma_1\}$ proves $\forall x.\phi_1(x)$ by necessarily non-analytic induction.

**

To more systematically settle conjectures like the one above one could attempt to establish more general results—instead of hand-crafting countermodels for each particular case. We hope that the literature on first-order arithmetic already contains lots of applicable results.

**

Verifying provability in weak fragments of arithmetic is hard. It is easy to rely on something true that is not provable in the fragment one works with. Thus it would be worthwhile to verify the provability statements in §4 with a theorem prover.

**

An interesting future line of work would be to consider other settings than arithmetic. For example, in computer science, many basic facts of functions on inductive structures seem to require non-analytic induction proofs.

**

One might also approach the problem of non-analytic induction proofs from the more proof-theoretical side, for example by studying derivations in natural deduction. Dag Prawitz's (2018) recent work may be useful.

**

§A Some results

DEFINITIONS. Let p be a polynomial in the indeterminate X or let p be a Laurent polynomial in the indeterminates X and X^{-1} .

- The *degree* of p , notation $\deg(p)$, is the largest n such that the coefficient of X^n is non-zero.
- The *order* of p , notation $\text{ord}(p)$, is the smallest n such that the coefficient of X^n is non-zero.

**

Examples.

$$- \deg(X^3+4) = 3.$$

$$- \text{ord}(X^3+4) = 0.$$

$$- \deg(X^2+20X^{-5}) = 2.$$

$$- \text{ord}(X^2+20X^{-5}) = -5.$$

**

LEMMA. Let $R[X]^+$ be the non-negative part of a discretely ordered commutative ring of polynomials. If $f : R[X]^+ \rightarrow R[X]^+$ satisfies

$$f(p+1) = (p+2)^2 f(p)$$

for all non-constant polynomials p then $f(p) = 0$ for all non-constant polynomials p .

**

PROOF. Let p be a non-constant polynomial. Note that if $f(p) \neq 0$ then $f(p-1) \neq 0$ and $\deg(f(p)) > \deg(f(p-1))$. Thus if $f(p) \neq 0$ for some non-constant polynomial p we have the contradiction that we have an infinitely descending chain of natural numbers:

$$\deg(f(p)) > \deg(f(p-1)) > \deg(f(p-2)) > \dots \quad \square$$

**

LEMMA. Let $R[X, X^{-1}]^+$ be the non-negative part of a discretely ordered commutative ring of Laurent polynomials. If $f : R[X, X^{-1}]^+ \rightarrow R[X, X^{-1}]^+$ satisfies

$$f(p+1) = (p+2)^2 f(p)$$

then $f(p) = 0$ for all non-constant polynomials p .

**

PROOF. Let p be a non-constant polynomial. As in the previous proof, note that if $f(p) \neq 0$ then $\deg(f(p)) > \deg(f(p-1))$. Also note that that $\text{ord}(f(p)) = \text{ord}(f(p-1))$. Thus if $f(p) \neq 0$ for some non-constant polynomial p we have the contradiction that for some natural number n we have

$$\deg(f(p-n)) < \text{ord}(f(p-n)). \quad \square$$

**

REMARK. We could have used the above proof for the previous lemma too.

**

FACT. Let M be an L_2 -structure that is the non-negative part of a discretely ordered commutative ring of polynomials, or of Laurent polynomials.

(1) If $M \models \text{DEF}(h)$ then $h^M(p) = 0$ for all non-constant polynomials p .

(2) If $M \models \text{DEF}(h) \wedge \text{DEF}(g)$ then $g^M(p) = 0$ for all non-constant polynomials p .

PROOF.

(1) The appropriate lemma is immediately applicable.

(2) Suppose $M \models \text{DEF}(h) \wedge \text{DEF}(g)$. Let p be a non-constant polynomial. We have

$$\begin{aligned} g^M(p+1) &= (p+2)^2 g^M(p) + h^M(p) && \text{(by } M \models \text{DEF}(g)\text{)} \\ &= (p+2)^2 g^M(p) && \text{(by (1) and } M \models \text{DEF}(h)\text{)}. \end{aligned}$$

Thus the appropriate lemma gives $g^M(p) = 0$ for all non-constant polynomials p . □

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REFERENCES

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