

NECESSARILY NON-ANALYTIC INDUCTION PROOFS—SUMMARY OF SOME RESULTS

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Contents

§1 Introduction.....	3
§2 Two results proved by non-analytic induction.....	5
§3 Tail-recursion.....	9
§4 Definitions.....	13
§5 Fact 2.2 must be proved by non-analytic induction.....	17
§6 Must Fact 2.4 be proved by non-analytic induction?.....	19
§7 Fact 3.2c must be proved by non-analytic induction.....	23
§8 Facts 3.2ab and 3.6 must be proved by non-analytic induction.....	29
§9 Discussion and future work.....	35
§A Some definitions and results.....	37
References.....	39

§1 Introduction

Some mathematical facts seem to not lend themselves to a "straightforward" induction proof—the proof attempt gets stuck at the induction step—but instead a "straightforward" induction proof of a "stronger" fact succeeds. This proof method is usually called something like "strengthening of the induction hypothesis". However, there need not always be a precise sense in which the "stronger" fact is, in fact, stronger. Thus, following Hetzl and Wong (2018), we use the more general terminology "non-analytic induction proofs" for such proofs. Consequently, we will use "analytic induction proofs" to refer to ordinary "straightforward" induction proofs.

**

Faced with a non-analytic induction proof, a natural question is whether the non-analyticity is necessary; that is, could we have proved what we wanted to prove by analytic induction, without the "detour" proving some other fact? Hetzl and Wong have made precise sense of such questions for first-order theories and sentences of arithmetic.

**

In §2 we give two examples of non-analytic induction proofs, proving the two facts:

(F1) For all natural numbers n : $1+3+5+\cdots+(2n-1) = k^2$ for some natural number k .

(F2) For all natural numbers n : $1+1/2^2+1/3^2+\cdots+1/(n+1)^2 < 2$.

**

In §3 we consider more examples of non-analytic induction proofs, by looking at correctness proofs for "tail-recursive" definitions of the factorial, exponentiation and of multiplication.

**

In §4 we present a slight generalization and reformulation of some of the notions introduced by Hetzl and Wong.

**

In §5 we show, using the notions introduced in §4, that there is a precise sense in which (F1) must be proved by non-analytic induction.

**

In §6 we present some preliminary results towards settling whether (F2) must be proved by non-analytic induction in any precise sense afforded by the notions introduced in §4.

**

In §7 and §8 we show, using the notions introduced §4, that there is a sense in which each correctness proof from §3 must use non-analytic induction.

**

In §9 we discuss the results and present some ideas for future work.

**

The lazy, or busy, reader may wish to skip §§6-8, as the core ideas of §§6-8 are present in simpler form in §5. Then §3 may be skipped as well since it is a prerequisite only for §§7-8.

**

§2 Two results proved by non-analytic induction

For our first case, we have that the sum of any initial segment of the odd natural numbers is a perfect square, that is,

for all natural numbers n : $1+3+5+\dots+(2n-1) = k^2$ for some natural number k ,

or, avoiding ellipsis notation, the following.

**

DEFINITION 2.1. $f_2 : \mathbb{N} \rightarrow \mathbb{N}$ is recursively defined by

$$f_2(0) := 0,$$

$$f_2(n+1) := f_2(n)+2n+1.$$

**

FACT 2.2. For all natural numbers n : $f_2(n) = k^2$ for some natural number k .

PROOF ATTEMPT. "Straightforward induction":

- Base case: We have

$$\begin{aligned} f_2(0) &= 0 \quad (\text{by Definition 2.1}) \\ &= 0^2 \end{aligned}$$

so $f_2(0) = k^2$ for $k = 0$.

- Induction step:

$$\begin{aligned} f_2(n+1) &= f_2(n)+2n+1 \quad (\text{by Definition 2.1}) \\ &= k^2+2n+1 \quad (\text{for some } k, \text{ by induction hypothesis}). \end{aligned}$$

But k^2+2n+1 is not a perfect square for all natural numbers k and n so how do we proceed from here?

ACTUAL PROOF. It suffices to prove the following stronger fact, of which Fact 2.2 is a logical consequence.

$f_2(n) = n^2$ for all natural numbers n .

(This fact is not a logical consequence of Fact 2.2, so it is stronger than Fact 2.2 in at least that sense.)

We prove this fact by induction.

- Base case:

$$\begin{aligned} f_2(0) &= 0 \quad (\text{by Definition 2.1}) \\ &= 0^2. \end{aligned}$$

- Induction step:

$$\begin{aligned} f_2(n+1) &= f_2(n)+2n+1 \quad (\text{by Definition 2.1}) \\ &= n^2+2n+1 \quad (\text{by induction hypothesis}) \\ &= (n+1)^2 \quad (\text{by elementary arithmetic}). \end{aligned}$$

□

**

For our second case, we have

$$1 + 1/2^2 + 1/3^2 + \dots + 1/(n+1)^2 < 2 \text{ for all natural numbers } n,$$

or, avoiding ellipsis notation, the following.

**

DEFINITION 2.3. $g_2 : \mathbb{N} \rightarrow \mathbb{Q}$ is recursively defined by

$$g_2(0) := 1,$$

$$g_2(n+1) := g_2(n) + 1/(n+2)^2.$$

**

FACT 2.4. $g_2(n) < 2$ for all natural numbers n .

PROOF ATTEMPT. "Straightforward" induction:

- Base case:

$$\begin{aligned} g_2(0) &= 1 && \text{(by Definition 2.3)} \\ &< 2. \end{aligned}$$

- Induction step:

$$\begin{aligned} g_2(n+1) &= g_2(n) + 1/(n+2)^2 && \text{(by Definition 2.3)} \\ &< 2 + 1/(n+2)^2 && \text{(by induction hypothesis)}. \end{aligned}$$

But $2 + 1/(n+2)^2 \not< 2$ for any natural number n so how do we proceed from here?

ACTUAL PROOF. It clearly suffices to prove the stronger fact

$$g_2(n) \leq 2 - 1/(n+1) \text{ for all natural numbers } n.$$

(In what sense is this fact stronger than Fact 2.4? It is at least stronger in the sense that for all $g : \mathbb{N} \rightarrow \mathbb{Q}$,

$$g(n) \leq 2 - 1/(n+1) \text{ for all } n$$

implies

$$g(n) < 2 \text{ for all } n,$$

while the converse implication does not hold for all $g : \mathbb{N} \rightarrow \mathbb{Q}$.)

We prove this fact by induction.

- Base case:

$$\begin{aligned} g_2(0) &= 1 && \text{(by Definition 2.3)} \\ &\leq 1 \\ &= 2 - 1/(0+1). \end{aligned}$$

- Induction step:

$$\begin{aligned}g_2(n+1) &= g_2(n) + 1/(n+2)^2 && \text{(by Definition 2.3)} \\ &\leq 2 - 1/(n+1) + 1/(n+2)^2 && \text{(by induction hypothesis)} \\ &= 2 - 1/(n+2) - 1/(n+1)(n+2)^2 && \text{(by lots of elementary arithmetic)} \\ &\leq 2 - 1/(n+2) && \text{(by more or less obvious facts). } \square\end{aligned}$$

**

§3 Tail-recursion

For practical reasons one might want to implement recursive functions such that they are "tail-recursive". We are not overly familiar with "tail-recursion" but for present purposes what matters is that we have non-standard "tail-recursive" definitions of common functions. We will consider tail-recursive definitions of the factorial, exponentiation and of multiplication. For each definition one naturally desires a correctness proof. It turns out that these correctness proofs seem to require non-analytic induction.

**

The technique we will use to convert the standard definitions to tail-recursive definitions is to introduce an extra "accumulator" argument.

**

DEFINITIONS 3.1.

(a) Multiplication $\cdot : \mathbb{N}^2 \rightarrow \mathbb{N}$ is defined recursively by

$$\begin{aligned} n \cdot 0 &:= 0, \\ n \cdot (m+1) &:= n \cdot m + n. \end{aligned}$$

(b) Exponentiation $-^{\cdot} : \mathbb{N}^2 \rightarrow \mathbb{N}$ is defined recursively by

$$\begin{aligned} n^0 &:= 1, \\ n^{m+1} &:= n^m \cdot n. \end{aligned}$$

(c) The factorial $! : \mathbb{N} \rightarrow \mathbb{N}$ is defined recursively by

$$\begin{aligned} 0! &:= 1, \\ (n+1)! &:= (n+1) \cdot n!. \end{aligned}$$

(d) The function $m\text{-acc} : \mathbb{N}^3 \rightarrow \mathbb{N}$ is defined recursively by

$$\begin{aligned} m\text{-acc}(a, n, 0) &:= a, \\ m\text{-acc}(a, n, m+1) &:= m\text{-acc}(a+n, n, m), \end{aligned}$$

(e) The function $e\text{-acc} : \mathbb{N}^3 \rightarrow \mathbb{N}$ is defined recursively by

$$\begin{aligned} e\text{-acc}(a, n, 0) &:= a, \\ e\text{-acc}(a, n, m+1) &:= e\text{-acc}(a \cdot n, n, m), \end{aligned}$$

(e) The function $f\text{-acc} : \mathbb{N}^2 \rightarrow \mathbb{N}$ is defined recursively by

$$\begin{aligned} f\text{-acc}(a, 0) &:= a, \\ f\text{-acc}(a, m+1) &:= f\text{-acc}(a \cdot (m+1), m). \end{aligned}$$

**

FACTS 3.2. For all natural numbers n and m :

$$(a) \ n \cdot m = m\text{-acc}(0, n, m),$$

$$(b) \ n^m = e\text{-acc}(1, n, m),$$

$$(c) \ n! = f\text{-acc}(1, n).$$

PROOF ATTEMPT. We try to prove (a) by induction on m . The base case is immediate. For the induction step we have

$$\begin{aligned} n \cdot (m+1) &= n \cdot m + n && \text{(by Definition 3.1a)} \\ &= m\text{-acc}(0, n, m) + n && \text{(by induction hypothesis)} \end{aligned}$$

and

$$\begin{aligned} m\text{-acc}(0, n, m+1) &= m\text{-acc}(0+n, n, m) && \text{(by Definition 3.1d)} \\ &= n\text{-acc}(n, n, m). \end{aligned}$$

It seems we cannot get any further and that we are "stuck". Similarly one gets "stuck" when attempting straightforward induction proofs of (b) and (c).

ACTUAL PROOF. (a)-(c) are immediate consequences of the following stronger facts. For all natural numbers a , n and m :

$$(a') \ m\text{-acc}(a, n, m) = a + n \cdot m,$$

$$(b') \ e\text{-acc}(a, n, m) = a \cdot n^m,$$

$$(c') \ f\text{-acc}(a, n) = a \cdot n!.$$

We prove (a'). The proof of (c') is similar and the proof of (b') is exactly the same up to substitution of function symbols and constants. For (a'), induction on m gives the following.

- Base case:

$$\begin{aligned} m\text{-acc}(a, n, 0) &= a && \text{(by Definition 3.1d)} \\ &= a + 0 \\ &= a + n \cdot 0 && \text{(by Definition 3.1a)}. \end{aligned}$$

- Induction step:

$$\begin{aligned} m\text{-acc}(a, n, m+1) &= m\text{-acc}(a+n, n, m) && \text{(by Definition 3.1d)} \\ &= (a+n) + n \cdot m && \text{(by induction hypothesis)} \\ &= a + (n \cdot m + n) && \text{(by associativity and commutativity)} \\ &= a + n \cdot (m+1) && \text{(by Definition 3.1a)}. \quad \square \end{aligned}$$

**

The above proof uses associativity and commutativity and additive identity of 0. These properties are in fact not necessary. We state and prove a more general fact as follows.

**

ASSUMPTION 3.3.

- Let A be any set.
- Let a_0 be an element in A .
- Let $*$: $A^2 \rightarrow A$.

**

DEFINITION 3.4.

- (a) The function $**$: $A \times \mathbb{N} \rightarrow A$ is defined recursively by

$$a**0 := a_0,$$

$$a**(n+1) := (a**n)*a.$$

- (b) The function acc : $A^2 \times \mathbb{N} \rightarrow A$ is defined recursively by

$$\text{acc}(a,b,0) := a,$$

$$\text{acc}(a,b,n+1) := \text{acc}(a*b,b,n).$$

**

LEMMA 3.5. For all a and b in A and for all natural numbers n :

$$\text{acc}(a,b,n)*b = \text{acc}(a*b,b,n).$$

PROOF. Induction on n .

- Base case:

$$\begin{aligned} \text{acc}(a,b,0)*b &= a*b && \text{(by Definition 3.4b)} \\ &= \text{acc}(a*b,b,0) && \text{(by Definition 3.4b)}. \end{aligned}$$

- Induction step:

$$\begin{aligned} \text{acc}(a,b,n+1)*b &= \text{acc}(a*b,b,n)*b && \text{(by Definition 3.4b)} \\ &= \text{acc}((a*b)*b,b,n) && \text{(by induction hypothesis)} \\ &= \text{acc}(a*b,b,n+1) && \text{(by Definition 3.4b)}. \end{aligned} \quad \square$$

**

FACT 3.6. For all a in A and all natural numbers n :

$$a**n = \text{acc}(a_0,a,n).$$

PROOF. With Lemma 3.5 at our disposal, straightforward induction on n will work.

- Base case:

$$\begin{aligned} a**0 &= a_0 && \text{(by Definition 3.4a)} \\ &= \text{acc}(a_0,a,0) && \text{(by Definition 3.4b)}. \end{aligned}$$

- Induction step:

$$\begin{aligned} a^{**}(n+1) &= (a^{**}n)*a && \text{(by Definition 3.4a)} \\ &= \text{acc}(a_0, a, n)*a && \text{(by induction hypothesis)} \\ &= \text{acc}(a_0*a, a, n) && \text{(by Lemma 3.5)} \\ &= \text{acc}(a_0, a, n+1) && \text{(by Definition 3.4b).} \quad \square \end{aligned}$$

**

REMARK. The above proof is an analytic induction proof. However, as is shown in §8, without first proving, by induction, something like Lemma 3.5, an analytic induction proof is not possible.

**

§4 Definitions

What follows in this section is a reformulation and slight generalization of some of the notions introduced by Hetzl and Wong (2017). We will introduce some non-conventional machinery, using rich languages in order to be able to conveniently and transparently represent mathematical facts and proofs. An alternative would be to choose one of the usual languages for Peano arithmetic and then use coding and representing formulas. We think our alternative is preferable since the task at hand involves modeling actual facts and proofs found in the wild. This modeling is straightforward in sufficiently rich languages. Furthermore, formulating our results using this machinery will allow us to draw interesting conclusions and to state interesting conjectures (see §9).

**

DEFINITIONS 4.1.

- The **full (first-order) language of arithmetic**, notation $\mathcal{L}[\text{full}]$, is the first-order language that for each natural number n has
 - a constant symbol n ,
 - a function symbol f of arity $n+1$ for each function $f : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$,
 - a relation symbol P of arity $n+1$ for each relation $P \subseteq \mathbb{N}^{n+1}$.
- The **minimal (first-order) language of arithmetic**, notation $\mathcal{L}[\text{min}]$, is the $\mathcal{L}[\text{full}]$ -reduct with signature $\langle 0, 1, + \rangle$.
- The **standard (first-order) language of arithmetic**, notation \mathcal{L}_0 , is the $\mathcal{L}[\text{full}]$ -reduct with signature $\langle 0, 1, +, \cdot, < \rangle$.
- A first-order language is a **(first-order) language of arithmetic** if and only if it is an $\mathcal{L}[\text{min}]$ -expansion.
- Let L be a language of arithmetic.
 - If L is an $\mathcal{L}[\text{full}]$ -reduct then L is **pure**; otherwise L is **impure**.
 - $\text{pure}(L) := L \cap \mathcal{L}[\text{full}]$ is the **pure part** of L .
 - $\text{impure}(L) := L - \mathcal{L}[\text{full}]$ is the **impure part** of L .

**

Thus any pure language of arithmetic has as symbols natural numbers and operations and relations on natural numbers. This is just a convenient "trick" which allows us to easily define the standard model for these languages.

**

DEFINITIONS 4.2.

- Let L be a pure language of arithmetic.
 - The **standard L-model** has domain \mathbb{N} and each symbol interpreted as itself.

- Let M be an L -model. If M is isomorphic to the standard L -model then M is **standard**; otherwise M is **non-standard**.
- The L -theory **true L -arithmetic** is the theory of the standard L -model.
- Any subset of true L -arithmetic is a **theory of L -arithmetic**.
- Let L be an impure language of arithmetic, let M be an L -model and let T be an L -theory.
 - M is **standard** if its pure(L)-reduct is; otherwise M is **non-standard**.
 - T is a **theory of L -arithmetic** if and only if T is true in some standard L -model.

**

REMARKS. Let L be an impure language.

- There are infinitely many non-isomorphic standard L -models—one for each way of expanding the standard pure(L)-model to L . (Similarly, there are infinitely many L -models that are not elementary equivalent. Thus there are also infinitely many non-compatible theories of L -arithmetic.)
- For an L -theory T there may be none, one or many non-isomorphic standard models of T (and T is a theory of L -arithmetic if and only if there is at least one such standard model).

**

NOTATIONS.

- \mathcal{N}_0 is the standard \mathcal{L}_0 -model.
- $\mathcal{N}[\text{min}]$ is the standard $\mathcal{L}[\text{min}]$ -model.
- $\mathcal{N}[\text{full}]$ is the standard $\mathcal{L}[\text{full}]$ -model.
- T_0 is true \mathcal{L}_0 -arithmetic.
- $T[\text{min}]$ is true $\mathcal{L}[\text{min}]$ -arithmetic.
- $T[\text{full}]$ is true $\mathcal{L}[\text{full}]$ -arithmetic.

**

DEFINITION 4.3. Let L be a language of arithmetic and let $\varphi(x)$ be an L -formula in the free variable x . The **induction instance** for $\varphi(x)$ is the L -sentence

$$\text{IND}(\varphi) := \varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x. \varphi(x).$$

**

DEFINITION 4.4. Let L be a language of arithmetic and let T be an L -theory. Let $\varphi(x)$ be an L -formula in the free variable x . Say that T proves $\forall x.\varphi(x)$ by necessarily non-analytic induction* if and only if there is an L -formula $\psi(x)$ in the free variable x such that

- (1) $T, \text{IND}(\varphi) \not\vdash \forall x.\varphi(x)$,
- (2) $T \vdash \varphi(0)$,
- (3) $T \vdash \psi(0)$,
- (4) $T \vdash \forall x: \psi(x) \rightarrow \psi(x+1)$,
- (5) $T \vdash \forall x.\psi(x) \rightarrow \forall x.\varphi(x)$.

Under conditions (1)–(5) we also say that $\psi(x)$ witnesses that T proves $\forall x.\varphi(x)$ by necessarily non-analytic induction*.

**

REMARKS.

- The above definition does not require T to be a theory of arithmetic (that is, T could contain sentences that are false in all standard L -models). We will of course require T to be a theory of arithmetic when using this definition to make precise sense of the necessary non-analyticity of non-analytic induction proofs (of arithmetic) found in the wild. But we do not rule out that there might be interesting uses of the above definition with T not a theory of arithmetic.
- As we saw when proving Fact 3.6, an analytic induction proof may be possible once an appropriate lemma has been proved (by induction). Such cases could qualify as necessarily non-analytic induction proofs under Definition 4.4 (and indeed, the proof of Fact 3.6 does, as shown in §8). This is because for any two induction axioms there is always an induction axiom logically equivalent to their conjunction. Thus any proof along the lines of "induction proof of a lemma followed by an analytic induction proof" could be rewritten as a single non-analytic induction proof (but that proof would probably not look very "natural"). In light of this, perhaps we should rethink our terminology, since it might seem a bit inappropriate to call proofs such as the proof of Fact 3.6 for "non-analytic induction proofs".

**

An alternative way of phrasing condition (1), provided by the following lemma, might be more intuitive.

**

LEMMA 4.5. (1) and (2) are equivalent to (2') and

- (2') $T \not\vdash \forall x: \varphi(x) \rightarrow \varphi(x+1)$.

PROOF. The direction from left (that is, from (1) and (2)) to right (that is, to (2') and (2)) is immediate. The direction from right to left is an easy (but non-trivial) exercise in propositional logic. \square

**

REMARK. We chose (1) instead of (1') in the definition precisely because the direction (1) \Rightarrow (1') is immediate (given (2)) while the direction (1') \Rightarrow (1) (given (2)) is not.

**

When trying to prove that typical T and $\varphi(x)$ satisfy the above definition, the hardest part will be—as far as we can tell at least—to show condition (1) (or condition (1')). Therefore it makes sense to break out a lemma that captures the idea of using non-standard models to establish (1).

**

LEMMA 4.6.

- Let L be a language of arithmetic.
- Let T be an L -theory.
- Let $\varphi(x)$ be an L -formula in the free variable x .
- Suppose $T \vdash \varphi(0)$.
- Suppose $M \models \forall x.\varphi(x)$ for all standard L -models $M \models T$.

The following are then equivalent.

- $T, \text{IND}(\varphi) \not\vdash \forall x.\varphi(x)$.
- There is a non-standard L -model $M \models T$ and a non-standard number c in M such that $M \models \varphi(c)$ and $M \not\models \varphi(c+1)$.

PROOF. Since $T \vdash \varphi(0)$, Lemma 4.5 gives that

$$T, \text{IND}(\varphi) \not\vdash \forall x.\varphi(x)$$

is equivalent to

$$T \not\vdash \forall x: \varphi(x) \rightarrow \varphi(x+1),$$

which by soundness and completeness is equivalent to

$$\text{there is } M \models T \text{ and } c \text{ in } M \text{ such that } M \models \varphi(c) \text{ and } M \not\models \varphi(c+1)$$

which, since $\forall x.\varphi(x)$ is true in all standard models of T , is equivalent to

$$\text{there is nonstandard } M \models T \text{ and a non-standard number } c \text{ in } M \text{ such that } M \models \varphi(c) \text{ and } M \not\models \varphi(c+1).$$

□

**

§5 Fact 2.2 must be proved by non-analytic induction

DEFINITION 5.1. The language L_5 of arithmetic is $\mathcal{L}[\text{full}]$ expanded with the unary function symbol 'f'.

**

REMARK. L_5 is thus an impure language, with $\text{impure}(L_5) = \{f\}$ (and with $\text{pure}(L_5) = \mathcal{L}[\text{full}]$).

**

DEFINITION 5.2.

- The L_5 -sentence $\text{DEF}(f)$ is defined by

$$\text{DEF}(f) := f(0) = 0 \wedge \forall x. f(x+1) = f(x)+2x+1.$$

- The theory T_5 of L_5 -arithmetic is defined by

$$T_5 := T[\text{full}]+\text{DEF}(f).$$

- The L_5 -formulas $\phi_5(x)$ and $\psi_5(x)$ are defined by

$$\phi_5(x) := \exists y. f(x) = y^2,$$

$$\psi_5(x) := f(x) = x^2.$$

**

The reader who finds the following fact puzzling or even contradictory might want to jump ahead to the remark after the proof.

**

FACT 5.3. $\psi_5(x)$ witnesses that T_5 proves $\forall x.\phi_5(x)$ by necessarily non-analytic induction.

PROOF. We need to show (1)–(5) of Definition 4.4. (2)–(5) are easy. To show (1) we appeal to Lemma 4.6 and exhibit a non-standard L_5 -model $M \models T_5$ with an element a such that $M \models \phi_5(a)$ and $M \not\models \phi_5(a+1)$. The following construction is due to Matt Kaufmann (private communication).

Let $M_0 \models T[\text{full}]$ be a non-standard $\mathcal{L}[\text{full}]$ -model and let a be any non-standard natural number in M_0 . We expand M_0 to an L_5 -model M by defining the interpretation $f^M : M_0 \rightarrow M_0$ as follows.

$$f^M(n) := n^2 - 9a^2 \quad \text{if } n = 5a+z \text{ for some integer } z,$$

$$f^M(n) := n^2 \quad \text{otherwise.}$$

We need to verify that this definition is well-formed; that is, that $n^2 - 9a^2$ is never negative when $n = 5a+z$ for some integer z . We have

$$\begin{aligned} f^M(5a+z) &= (5a+z)^2 - 9a^2 \\ &= 25a^2 + 10za + z^2 - 9a^2 \\ &= 16a^2 + 10za + z^2 \\ &= a(16a + 10z) + z^2 \end{aligned}$$

which is positive since

$$a > 0 \quad (\text{since } a \text{ is non-standard}),$$

$$16a+10z > 0 \quad (\text{since } a \text{ is non-standard and } z \text{ is not}),$$

$$z^2 \geq 0 \quad (\text{since } z^2 \text{ is a square}).$$

With f^M so defined it is easy to verify that $M \models T_5$. We have $M \models \varphi_5(5a)$ since

$$\begin{aligned} \varphi_5(5a) &= (5a)^2 - 16a^2 \\ &= 25a^2 - 16a^2 \\ &= 9a^2 \\ &= (3a)^2. \end{aligned}$$

It remains to show $M \not\models \varphi_5(5a+1)$. We have

$$\begin{aligned} \varphi_5(5a+1) &= (5a+1)^2 - 16a^2 \\ &= 25a^2 + 10a + 1 - 16a^2 \\ &= 9a^2 + 10a + 1. \end{aligned}$$

This is not a perfect square in M since

$$\forall x \forall y: x > 0 \rightarrow 9x^2 + 10x + 1 \neq y^2$$

is a theorem of $T[\text{full}]$. Thus we have $M \not\models \varphi_5(5a+1)$. □

**

REMARK. Fact 5.3 might seem contradictory. After all, there is a unique standard L_5 -model, namely $\mathbb{N}[\text{full}]$ with f interpreted by f_2 (from §2), and in this model $\forall x.\varphi(x)$ is of course true. The catch is that T_5 is weaker than the theory of this standard model, since T_5 does not contain any induction axioms involving the symbol f . In particular, T_5 does not prove that f is equal to f_2 (which, being a function $\mathbb{N} \rightarrow \mathbb{N}$, is a symbol of $\mathcal{L}[\text{full}]$ and thus also of L_5). For more discussion of this phenomenon see §9.

**

§6 Must Fact 2.4 be proved by non-analytic induction?

Since Fact 2.4 is a statement involving rationals, a little work is needed to phrase it as a natural statement in first-order arithmetic (that is, without involving any coding).

**

DEFINITION 6.1. $g_\epsilon : \mathbb{N} \rightarrow \mathbb{N}$ and $h_\epsilon : \mathbb{N} \rightarrow \mathbb{N}$ are recursively defined by

$$g_\epsilon(0) := 1,$$

$$g_\epsilon(n+1) := (n+2)^2 g_\epsilon(n) + h_\epsilon(n),$$

$$h_\epsilon(0) := 1,$$

$$h_\epsilon(n+1) := (n+2)^2 h_\epsilon(n).$$

**

By induction, we have

$$g_2(n) = g_\epsilon(n)/h_\epsilon(n)$$

so the inequality $g_2(n) < 2$ can be rewritten as

$$g_\epsilon(n) < 2h_\epsilon(n).$$

Similarly the inequality $g_2(n) \leq 2 - 1/(n+1)$ becomes

$$(n+1)g_\epsilon(n) \leq (2n+1)h_\epsilon(n).$$

**

All in all, we have the following rephrasings of Fact 2.4 and its proof attempt and proof.

**

LEMMA 6.2. For all natural numbers n , if $(n+1)g_\epsilon(n) \leq (2n+1)h_\epsilon(n)$ then $g_\epsilon(n) < 2h_\epsilon(n)$.

PROOF. Suppose $(n+1)g_\epsilon(n) \leq (2n+1)h_\epsilon(n)$. Since $(2n+1)h_\epsilon(n) < 2(n+1)h_\epsilon(n)$ we then have $(n+1)g_\epsilon(n) < 2(n+1)h_\epsilon(n)$ so we must have $g_\epsilon(n) < 2h_\epsilon(n)$. \square

**

FACT 6.3. $g_\epsilon(n) < 2h_\epsilon(n)$ for all n .

PROOF ATTEMPT. Induction on n .

- Base case:

$$\begin{aligned} g_\epsilon(0) &= 1 && \text{(by Definition 6.1)} \\ &< 2 \\ &= 2 \cdot 1 \\ &= 2 \cdot h_\epsilon(0) && \text{(by Definition 6.1)}. \end{aligned}$$

- Induction step:

$$\begin{aligned} g_\epsilon(n+1) &= (n+2)^2 g_\epsilon(n) + h_\epsilon(n) && \text{(by Definition 6.1)} \\ &< (n+2)^2 \cdot 2h_\epsilon(n) + h_\epsilon(n) && \text{(by induction hypothesis)} \end{aligned}$$

But for any n we have $(n+2)^2 \cdot 2h_\epsilon(n) + h_\epsilon(n) \not\leq 2h_\epsilon(n+1) = 2(n+2)^2 h_\epsilon(n)$. Thus we are "stuck".

ACTUAL PROOF. By Lemma 6.2, it suffices to prove

$$(n+1)g_\epsilon(n) \leq (2n+1)h_\epsilon(n) \text{ for all } n.$$

We prove this by induction on n .

- Base case:

$$\begin{aligned} (0+1)g_\epsilon(0) &= (0+1) \cdot 1 && \text{(by Definition 6.1)} \\ &= 1 \\ &\leq 1 \\ &= (2 \cdot 0 + 1) \cdot 1 \\ &= (2 \cdot 0 + 1)h_\epsilon(0) && \text{(by Definition 6.1)}. \end{aligned}$$

- Induction step: Our induction hypothesis is

$$(IH) \quad (n+1)g_\epsilon(n) \leq (2n+1)h_\epsilon(n)$$

and we must show

$$(GOAL) \quad (n+2)g_\epsilon(n+1) \leq (2n+3)h_\epsilon(n+1).$$

We have

$$\begin{aligned} (n+2)g_\epsilon(n+1) &= (n+2)((n+2)^2 g_\epsilon(n) + h_\epsilon(n)) && \text{(by Definition 6.1)} \\ &= (n+2)^3 g_\epsilon(n) + (n+2)h_\epsilon(n) \end{aligned}$$

and

$$\begin{aligned} (2n+3)h_\epsilon(n+1) &= (2n+3)(n+2)^2 h_\epsilon(n) && \text{(by Definition 6.1)} \\ &= (2n+3)(n+2)(n+2)h_\epsilon(n) \\ &= (2n^2+7n+6)(n+2)h_\epsilon(n) \\ &= (2n^2+7n+5)(n+2)h_\epsilon(n) + (n+2)h_\epsilon(n) \end{aligned}$$

so (GOAL) is equivalent to

$$(n+2)^3 g_\epsilon(n) \leq (2n^2+7n+5)(n+2)h_\epsilon(n),$$

which is equivalent to

$$(n+2)^2 g_\epsilon(n) \leq (2n^2+7n+5)h_\epsilon(n),$$

which we have by

$$\begin{aligned} (n+2)^2 g_\epsilon(n) &= ((n+1)+1)^2 g_\epsilon(n) \\ &= ((n+1)^2 + 2(n+1) + 1)g_\epsilon(n) \\ &= ((n+1)^2 + 2(n+1))g_\epsilon(n) + g_\epsilon(n) \\ &= (n+3)(n+1)g_\epsilon(n) + g_\epsilon(n) \\ &\leq (n+3)(2n+1)h_\epsilon(n) + g_\epsilon(n) && \text{(by (IH))} \\ &\leq (n+3)(2n+1)h_\epsilon(n) + 2h_\epsilon(n) && \text{(by Lemma 6.2 and (IH))} \\ &= (2n^2+7n+5)h_\epsilon(n). \end{aligned} \quad \square$$

**

REMARK. We find it a bit curious that the proof we gave of Fact 2.4 uses only one application of the induction hypothesis, while the above proof of Fact 6.3 uses two applications. Is there a proof of Fact 6.3 which, in some sensible precise sense, uses only one application of the induction hypothesis?

**

DEFINITION 6.4.

- The language of arithmetic L_6 is $\mathcal{L}[\text{full}]$ expanded with the unary function symbols 'g' and 'h'.

- The L_6 -sentences $\text{DEF}(g)$ and $\text{DEF}(h)$ are defined by

$$\text{DEF}(g) := g(0) = 1 \wedge \forall x. g(x+1) = (x+2)^2 \cdot g(x) + h(x),$$

$$\text{DEF}(h) := h(0) = 1 \wedge \forall x. h(x+1) = (x+2)^2 \cdot h(x).$$

- The theory T_6 of L_6 -arithmetic is defined by

$$T_6 := T[\text{full}] + \text{DEF}(g) + \text{DEF}(h).$$

- The L_6 -formulas $\phi_6(x)$ and $\psi_6(x)$ are defined by

$$\phi_6(x) := g(x) < 2 \cdot h(x),$$

$$\psi_6(x) := (x+1) \cdot g(x) + h(x) \leq 2 \cdot (x+1) \cdot h(x).$$

**

CONJECTURE 6.5. $\psi_6(x)$ witnesses that T_6 proves $\forall x. \phi_6(x)$ by necessarily non-analytic induction.

**

We might try to adapt the proof of the previous section: Cleverly interpret function symbols on a non-standard $\mathcal{L}[\text{full}]$ -model. Let M_0 be a non-standard $\mathcal{L}[\text{full}]$ -model. We want to extend M_0 to an L_6 -model $M \models T_6$ such that for some non-standard a in M we have

$$M \models \phi(a)$$

and

$$M \not\models \phi(a+1),$$

that is, we need an a in M_0 and interpretations $g^M, h^M : M_0 \rightarrow M_0$ such that

$$(a) \quad g^M(0) = h^M(0) = 1,$$

(b) for all n in M_0 :

$$- \quad g^M(n+1) = (n+2)^2 \cdot g^M(n) + h^M(n),$$

$$- \quad h^M(n+1) = (n+2)^2 \cdot h^M(n),$$

$$(c) \quad g^M(a) < 2 \cdot h^M(a),$$

$$(d) \quad g^M(a+1) \not\propto 2 \cdot h^M(a+1).$$

By (c) we must have $h^M(a) \neq 0$. Then we must have

$$\begin{aligned} h^M(a) &= h^M((a-1)+1) \\ &= (a+1)^2 \cdot h^M(a-1) && \text{(by (b))} \\ &= (a+1)^2 \cdot h^M((a-2)+1) \\ &= (a+1)^2 \cdot a^2 \cdot h^M(a-2) && \text{(by (b))} \\ &= (a+1)^2 \cdot a^2 \cdot (a-1)^2 \cdot h^M(a-3) && \text{(by (b))} \\ &= (a+1)^2 \cdot a^2 \cdot (a-1)^2 \cdot (a-2)^2 \cdot h^M(a-4) && \text{(by (b))} \\ &\vdots \end{aligned}$$

Thus it is necessary that $h^M(a)$ is non-zero and divisible by each of

$$(a+1)^2, a^2, (a-1)^2, (a-2)^2, \dots$$

We get similar requirements on g^M . We guess that if it is possible to construct a non-standard L_6 -model M with interpretations g^M and h^M satisfying (a)-(d), then this construction would be easy for experts on models of arithmetic. We need further familiarization with the literature, however.

**

§7 Fact 3.2c must be proved by non-analytic induction

DEFINITIONS 7.1.

- The language L_7 of arithmetic is $\mathcal{L}[\text{full}]$ expanded with the binary function symbol 'acc₇'.

- The L_7 -sentence $\text{DEF}(\text{acc}_7)$ is defined by

$$\text{DEF}(\text{acc}_7) := \forall a. \text{acc}_7(a,0) = a \wedge \forall a \forall x. \text{acc}_7(a,x+1) = \text{acc}_7(a \cdot (x+1),x).$$

- The theory T_7 of L_7 -arithmetic is defined by

$$T_7 := T[\text{full}] + \text{DEF}(\text{acc}_7).$$

- The L_7 -formulas $\phi_7(x)$ and $\psi_7(x)$ are defined by

$$\phi_7(x) := x! = \text{acc}_7(1,x),$$

$$\psi_7(x) := \forall a. \text{acc}_7(a,x) = a \cdot x!.$$

**

FACT 7.2. $\psi_7(x)$ witnesses that T_7 proves $\forall x. \phi_7(x)$ by necessarily non-analytic induction.

**

To prove Fact 7.2 we will proceed as in the proof of Fact 5.3.

Conditions (2)-(5) of Definition 4.4 are straightforward. To prove (1) we take any non-standard model $M_0 \models T[\text{full}]$ and expand it to an L_7 -model M , interpreting acc_7 such that we get $M \models T_7$ and a non-standard c such that $M \models \phi_7(c)$ and $M \not\models \phi_7(c+1)$.

**

ASSUMPTIONS 7.3.

- Let $M_0 \models T[\text{full}]$ be a non-standard $\mathcal{L}[\text{full}]$ -model.

- Let c be a non-standard number in M_0 .

**

We will expand M_0 to an L_7 -model $M \models T_7$ such that $M \models \phi_7(c)$ and $M \not\models \phi_7(c+1)$. The defining equations for acc_7 (that is, the sentence $\text{DEF}(\text{acc}_7)$) fix all its interpretations $\text{acc}_7^M(a,n)$ where n is standard. For non-standard n we have quite some leeway. To achieve $M \models \phi_7(c)$ and $M \not\models \phi_7(c+1)$ we will put

$$\text{acc}_7^M(1,c) := c!$$

and

$$\text{acc}_7^M(1,c+1) := 0.$$

These together with the second conjunct of $\text{DEF}(\text{acc}_7)$ then gives

$$\begin{aligned} c! &= \text{acc}_7^M(1,c) \\ &= \text{acc}_7^M(c,c-1) \\ &= \text{acc}_7^M(c(c-1),c-2) \\ &= \text{acc}_7^M(c(c-1)(c-2),c-3) \\ &\vdots \end{aligned}$$

respectively

$$\begin{aligned} 0 &= \text{acc}_7^M(1,c+1), \\ &= \text{acc}_7^M(c+1,c), \\ &= \text{acc}_7^M((c+1)c,c-1), \\ &= \text{acc}_7^M((c+1)c(c-1),c-2), \\ &= \text{acc}_7^M((c+1)c(c-1)(c-2),c-3), \\ &\vdots \end{aligned}$$

The defining equations impose no further restrictions so we can put $\text{acc}_7^M(n,m) := 0$ for all yet to be defined cases.

**

We will now make the construction in the preceding paragraph more precise.

**

DEFINITIONS 7.4.

- The set $A \subseteq M_0^2$ is defined by

$$A := \{(a,n) : n \text{ is standard}\},$$

- The set $B \subseteq M_0^2$ is defined inductively by

$$\frac{}{(1,c) \text{ is in } B} \quad \frac{(a,n+1) \text{ is in } B}{(a(n+1),n) \text{ is in } B} .$$

- The set $C \subseteq M_0^2$ is defined by

$$C := M_0^2 - (A \cup B).$$

**

REMARK. The elements of B are thus

$$\begin{aligned} &(1,c), \\ &(c,c-1), \\ &(c(c-1),c-2), \\ &(c(c-1)(c-2),c-3), \\ &\vdots \end{aligned}$$

**

LEMMA 7.5. A, B and C partitions M_0^2 .

PROOF. Clearly:

- $A \cup B \cup C = M_0^2$.
- C is disjoint from A and from B .

It thus remains to show that A and B are disjoint. If (a, n) is in A then n is standard but if (a, n) is in B then n is non-standard. Thus A and B have no common elements and thus they are disjoint. \square

**

By Lemma 7.5 the following definition is well-formed.

**

DEFINITION 7.6. $\text{acc}_7^M : M_0^2 \rightarrow M_0$ is defined by

$$\begin{aligned} \text{acc}_7^M(a, n) &:= a \cdot n! && \text{if } n \text{ is standard,} \\ \text{acc}_7^M(a, n) &:= c! && \text{if } (a, n) \text{ is in } B, \\ \text{acc}_7^M(a, n) &:= 0 && \text{if } (a, n) \text{ is in } C. \end{aligned}$$

**

LEMMA 7.7. Let X be A, B or C . For all a and n in M_0 , $(a, n+1)$ is in X if and only if $(a(n+1), n)$ is.

PROOF. It suffices to show this for $X = A$ and $X = B$ (since then it must hold for C , by definition of C).

- Case $X = A$:

$$\begin{aligned} (a, n+1) \text{ is in } A & \text{ iff } n+1 \text{ is standard} \\ & \text{ iff } n \text{ is standard} \\ & \text{ iff } (a(n+1), n) \text{ is in } A. \end{aligned}$$

- Case $X = B$: The 'only if' direction holds by definition. For the 'if' direction, suppose $(a(n+1), n)$ is in B . Case splitting on the inductive definition of B gives the following.

- Case $(a(n+1), n) = (1, c)$: Then $n = c$ and $a(n+1) = 1$, so $a(c+1) = 1$ so $a = c+1 = 1$, so $c = 0$. Since c is non-standard this is a contradiction so this case is not possible.

- Case $(a(n+1), n) = (a'(n'+1), n')$ for some a' and n' in M_0 such that $(a', n'+1)$ is in B : Then $a(n+1) = a'(n'+1)$ and $n = n'$ so $a(n+1) = a'(n+1)$ so $a = a'$ (by the appropriate well-known theorem of $T[\text{full}]$). Thus $(a, n+1) = (a', n'+1)$ and since $(a', n'+1)$ is in B so is $(a, n+1)$. \square

**

LEMMA 7.8. For all a and n in M_0 , if (a,n) is in B then $n \leq c$.

PROOF. Straightforward induction on " (a,n) is in B ".

- Case $(a,n) = (1,c)$: Immediate.

- Case $(a,n) = (a'(n+1),n)$ for some a' in M_0 and with induction hypothesis $n+1 \leq c$: Immediate. \square

**

LEMMA 7.9.

(1) $M \models T_7$.

(2) $M \models \phi_7(c)$.

(3) $M \not\models \phi_7(c+1)$.

PROOF.

(1) We need to show:

(a) $M \models T[\text{full}]$,

(b) $M \models \forall a. \text{acc}_7(a,0) = a$,

(c) $M \models \forall a \forall x. \text{acc}_7(a,x+1) = \text{acc}_7(a \cdot (x+1),x)$.

(a) holds since M is an M_0 -expansion. (b) holds by Definition 7.6. For (c), suppose a and n are in M_0 . We need to show

(*) $\text{acc}_7^M(a,n+1) = \text{acc}_7^M(a(n+1),n)$.

By Lemma 7.5 it suffices to consider the following cases.

- Case $(a,n+1)$ is in A :

$$\begin{aligned} \text{acc}_7^M(a,n+1) &= a \cdot (n+1)! && \text{(by Definition 7.6)} \\ &= a \cdot ((n+1) \cdot n!) && \text{(by Definition 3.1c)} \\ &= (a \cdot (n+1)) \cdot n! && \text{(by associativity)} \\ &= \text{acc}_7^M(a(n+1),n) && \text{(by Definition 7.6)} \end{aligned}$$

- Case $(a,n+1)$ is in B : Then, by Lemma 7.7, $(a(n+1),n)$ is in B as well so, by Definition 7.6, both sides of (*) are $c!$ and thus (*) holds.

- Case $(a,n+1)$ is in C : Similar to the previous case.

(2) We need to show

$$c! = \text{acc}_7^M(1,c).$$

Since $(1,c)$ is in B , this holds by Definition 7.6.

(3) We need to show

(*) $(c+1)! \neq \text{acc}_7^M(1,c+1)$.

$c+1$ is non-standard so $(1,c+1)$ is not in A . We have $c+1 > c$, so by Lemma 7.8, $(1,c+1)$ is not in B . Thus $(1,c+1)$ is in C and thus, by

Definition 7.6, we have $\text{acc}_M(1, c+1) = 0$. Thus (*) holds since $M \models T[\text{full}]$ and

$$\forall x. x! \neq 0$$

is a theorem of $T[\text{full}]$.

**

PROOF OF FACT 7.2. We need to show (1)-(5) of Definition 4.4. (1) follows from Lemmas 4.6 and 7.9. (2)-(5) are easy to verify. \square

**

§8 Facts 3.2ab and 3.6 must be proved by non-analytic induction

In this section we present a general result which shows that facts like Facts 3.2ab must be proved by non-analytic induction. Since Fact 3.6 is a generalization of Facts 3.2ab, this means that Fact 3.6 too must be proved by non-analytic induction.

**

We now reuse '*' and '**'. They were used in §3 but those usages does thus not apply anymore.

**

ASSUMPTIONS 8.1.

- Let $*$: $\mathbb{N}^2 \rightarrow \mathbb{N}$.
- Let n_0 be a natural number.
- Let $M_0 \models T[\text{full}]$ be non-standard $\mathcal{L}[\text{full}]$ -model.
- Let c be a non-standard number in M_0 .

**

REMARK. In §3 we had the more general $*$: $A \times A \rightarrow A$, with A an arbitrary set. Since the notions introduced in §4 only covers arithmetic we now consider only the case $A = \mathbb{N}$, although there should be no problem generalizing to a more general setting (once the notions in §4 have been appropriately generalized).

**

DEFINITION 8.2. $**$: $\mathbb{N}^2 \rightarrow \mathbb{N}$ is defined recursively by

$$\begin{aligned} n^{**0} &:= n_0, \\ n^{**(m+1)} &:= (n^{**m}) * n. \end{aligned}$$

**

DEFINITIONS 8.3.

- The language L_δ of arithmetic is $\mathcal{L}[\text{full}]$ expanded with the ternary function symbol ' acc_δ '.
- The L_δ -sentence $\text{DEF}(\text{acc}_\delta)$ is defined by

$$\begin{aligned} \text{DEF}(\text{acc}_\delta) &:= \\ &\forall a \forall x. \text{acc}_\delta(a, x, 0) = a \wedge \forall a \forall x \forall y. \text{acc}_\delta(a, x, y+1) = \text{acc}_\delta(a * x, x, y). \end{aligned}$$

- The theory T_δ of L_δ -arithmetic is defined by

$$T_\delta := T[\text{full}] + \text{DEF}(\text{acc}_\delta).$$

- The L_δ -formula $\phi_\delta(y)$ is defined by

$$\phi_\delta(y) := \forall x. \text{acc}_\delta(n_0, x, y) = x^{**y}.$$

**

Proceeding similarly to §5 and §7, we will expand M_0 to an L_δ -model $M \models T_\delta$ such that $M \models \varphi(c-1)$ and $M \not\models \varphi(c)$. To achieve $M \models \varphi(c-1)$ we will put

$$\text{acc}_\delta^M(n_0, n, c-1) := n^{**}(c-1)$$

for all n in M_0 . To achieve $M \not\models \varphi(c)$ we will put

$$\text{acc}_\delta^M(n_0, n, c) := (n^{**}c)+1$$

for some n in M_0 . Now whether we can do this while satisfying $\text{DEF}(\text{acc}_\delta)$ depends on $*$ and n_0 . For example, for $*$ = + and $n_0 = 0$, we would put

$$\text{acc}_\delta^M(0, n, c-1) := n \cdot (c-1).$$

To satisfy the second conjunct of $\text{DEF}(\text{acc}_\delta)$ we would then need, for all n in M_0 ,

$$\begin{aligned} n \cdot (c-1) &= \text{acc}_\delta^M(0, n, c-1) \\ &= \text{acc}_\delta^M(n, n, c-2) \\ &= \text{acc}_\delta^M(2n, n, c-3) \\ &= \text{acc}_\delta^M(3n, n, c-4) \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} 0 &= 0 \cdot (c-1) \\ &= \text{acc}_\delta^M(0, 0, c-1) \\ &= \text{acc}_\delta^M(0, 0, c) \\ &= \text{acc}_\delta^M(0, 0, c+1) \\ &= \text{acc}_\delta^M(0, 0, c+2) \\ &\vdots \end{aligned}$$

We will now make the construction more precise.

**

DEFINITIONS 8.4.

- The set $A \subseteq M_0^3$ is defined by

$$A := \{(a, n, m) : m \text{ is standard}\},$$

- The set $B \subseteq M_0^3$ is defined inductively by

$$\frac{n \text{ is in } M_0}{(n_0, n, c-1) \text{ is in } B} \quad \frac{(a, n, m+1) \text{ is in } B}{(a^*n, n, m) \text{ is in } B} \quad \frac{(a^*n, n, m) \text{ is in } B}{(a, n, m+1) \text{ is in } B} .$$

- The set $C \subseteq M_0^3$ is defined by

$$C := M_0^3 - (A \cup B).$$

**

EXAMPLES 8.5.

(a) If $*$ = + and $n_0 = 0$ then B consists of the triples

(0,0,c),
 (0,0,c+1),
 (0,0,c+2),
 :

together with all triples that are on one of the forms

(0,n,c-1),
 (n,n,c-2),
 (2n,n,c-3),
 (3n,n,c-4),
 :

(b) If $*$ = \cdot and $n_0 = 1$ then B consists of the triples

(1,1,c),
 (1,1,c+1),
 (1,1,c+2),
 :

together with all triples that are on one of the forms

(1,n,c-1),
 (n,n,c-2),
 (n^2 ,n,c-3),
 (n^3 ,n,c-4),
 :

**

LEMMA 8.6. A, B and C partitions M_0^3 .

PROOF. Clearly:

- $A \cup B \cup C = M_0^3$.
- C is disjoint from A and from B.

It thus remains to show that A and B are disjoint. If (a,n,m) is in A then m is standard but if (a,n,m) is in B then n is non-standard. Thus A and B have no common elements and thus they are disjoint. \square

**

By Lemma 8.6, the following definition is well-formed.

**

DEFINITION 8.7. $\text{acc}_0^M : M_0^3 \rightarrow M_0$ is defined by

$$\begin{aligned} \text{acc}_0^M(a,n,0) &:= a, \\ \text{acc}_0^M(a,n,m+1) &:= \text{acc}_0^M(a*n,n,m) \quad \text{if } m \text{ is standard,} \\ \text{acc}_0^M(a,n,m) &:= n**(c-1) \quad \text{if } (a,n,m) \text{ is in B,} \\ \text{acc}_0^M(a,n,m) &:= (n**c)+1 \quad \text{if } (a,n,m) \text{ is in C.} \end{aligned}$$

**

LEMMA 8.8. Let X be A , B or C . For all a , n and m in M_0 , $(a,n,m+1)$ is in X if and only if $(a*n,n,m)$ is.

PROOF. It suffices to show this for $X = A$ and $X = B$ (since then it must hold for C , by definition of C). For $X = B$ this holds by definition. For $X = A$:

$$\begin{aligned} (a,n,m+1) \text{ is in } A &\text{ iff } m+1 \text{ is standard} \\ &\text{iff } m \text{ is standard} \\ &\text{iff } (a*n,n,m) \text{ is in } A. \end{aligned}$$

□

**

ASSUMPTION 8.9. There is n in M_0 such that (n_0,n,c) is in C .

**

LEMMA 8.10.

- (1) $M \models T_0$.
- (2) $M \models \varphi_0(c-1)$.
- (3) $M \not\models \varphi_0(c)$.

PROOF.

(1) We need to show:

- (a) $M \models T[\text{full}]$,
- (b) $M \models \forall a \forall x. \text{acc}_0(a,x,0) = a$,
- (c) $M \models \forall a \forall x \forall y. \text{acc}_0(a,x,y+1) = \text{acc}_0(a*x,x,y)$.

(a) holds since M is an M_0 -expansion. (b) holds by Definition 8.6. For (c), suppose a , n and m are in M_0 . We need to show

$$(*) \text{acc}_0^M(a,n,m+1) = \text{acc}_0^M(a*n,n,m).$$

By Lemma 8.6 it suffices to consider the following cases.

- Case $(a,n,m+1)$ is in A : Then $(*)$ holds by Definition 8.7.
- Case $(a,n,m+1)$ is in B : Then, by Lemma 8.8, $(a*n,n,m)$ is in B as well. Thus, by Definition 8.7, both sides of $(*)$ are equal.
- Case $(a,n,m+1)$ is in C : Similar to the previous case.

(2) We need to show that

$$\text{acc}_0^M(n_0, n, c-1) = n^{c-1}$$

for all n in M_0 . Since $(n_0, n, c-1)$ is in B for all n in M_0 , this holds by Definition 8.7.

(3) We need to exhibit an n in M_0 such that

$$\text{acc}_0^M(n_0, n, c) \neq n^c.$$

By Assumption 8.9 we have an n in M_0 such that (n_0, n, c) is in C , so for this n we have

$$\begin{aligned} \text{acc}_0(n_0, n, c) &= (n^c)+1 && \text{(by Definition 8.7)} \\ &\neq n^c && \text{(by well-known theorem of T[full]).} \quad \square \end{aligned}$$

**

FACT 8.11. With $*$ = + and $n_0 = 0$, or with $*$ = \cdot and $n_0 = 1$, T_0 proves $\forall y. \varphi(y)$ by necessarily non-analytic induction.

PROOF. Both cases are similar. We prove the case $*$ = +. We must exhibit an L_0 -formula $\psi(y)$ such that conditions (1)-(5) of Definition 4.4 are satisfied. As the proof of Fact 3.2 shows,

$$\psi(y) := \forall a \forall x. \text{acc}_0(a, x, y) = a+x \cdot y$$

will work. Conditions (2)-(5) are then straightforward. Condition (1) follows from Lemmas 4.6 and 8.10. It remains to verify that Assumption 8.9 is true when $*$ = + and $n_0 = 0$, which it is since for example $(0, 1, c)$ is in C (see Example 8.5a). □

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§9 Discussion and future work

What does results like Facts 5.3, 7.2 and 8.11 tell us about the necessity of non-analytic induction in mathematical practice? Consider Fact 5.3 and consider a working mathematician who have just introduced the function $f_2 : \mathbb{N} \rightarrow \mathbb{N}$ by Definition 2.1—that is, by the recursive definition

$$f_2(0) := 0,$$

$$f_2(n+1) := f_2(n) + 2n + 1.$$

Fact 5.3 says that if our working mathematician wants to prove—by the means provided by first-order arithmetic—that $f_2(n)$ is a perfect square for all n , then no matter what facts have been previously established, a non-analytic induction proof is necessary (or put differently: a non-analytic induction axiom is needed). As Fact 5.3 shows, even if our working mathematician would somehow have established all first-order facts of arithmetic in a language without ' f_2 ', a non-analytic induction axiom would still be needed. Thus results like Fact 5.3 say that some non-analytic induction proofs are necessarily non-analytic in this rather strong sense.

**

PA^- , in the language $\mathcal{L}[\text{min}]$, is "the theory of the non-negative parts of discretely ordered commutative rings" (see §A). Hetzl and Wong (2017) consider the problem of finding non-analytic induction proofs from PA^- . This is motivated by the fact that for any $\mathcal{L}[\text{min}]$ -sentence σ , if $PA \vdash \sigma$ then there is an $\varphi(x)$ such that $PA^-, \text{IND}(\varphi) \vdash \sigma$. In previous work, we stated and proved Facts 5.3 and 7.2 with PA^- instead of $T[\text{full}]$ (in the definition of T_5 and T_7 , respectively). Naturally, one might wonder whether one could always replace PA^- with $T[\text{full}]$ in such results. This is not the case for trivial reasons: Facts formulated using only $\mathcal{L}[\text{min}]$ might need non-analytic induction axioms added to PA^- to be provable, but these non-analytic induction axioms are of course already in $T[\text{full}]$. Our Facts 5.3, 7.2 and 8.11 concerns results formulated in an impure language. Perhaps one could find some general criterion that says that for some class of statements in such languages, if $PA^- + T$ proves them by necessarily non-analytic induction then so does $T[\text{full}] + T$ (where T would typically consist of axioms governing the symbols from the impure part of the language). For example one could investigate the following conjecture, which is inspired by Facts 7.2 and 8.11.

**

CONJECTURE 9.1.

- Let L be $\mathcal{L}[\text{full}]$ expanded with the $(n+m)$ -ary function symbol 'f'.
- Let T be an L -theory of arithmetic.
- Let t_1, \dots, t_n be closed \mathcal{L}_0 -terms.
- Let $t(x_1, \dots, x_m)$ be an \mathcal{L}_0 -term.
- Suppose $\text{PA}^- + T$ proves

$$\sigma := \forall x_1 \cdots \forall x_m. t(x_1, \dots, x_m) = f(t_1, \dots, t_n, x_1, \dots, x_m).$$

by necessarily non-analytic induction.

Then $T[\text{full}] + T$ proves σ by necessarily non-analytic induction.

**

Another line of future work would of course be to settle Conjecture 6.5.

**

To more systematically settle conjectures like Conjecture 6.5 one could attempt to establish more general results—instead of hand-crafting countermodels for each particular case. This might be difficult. As a starting point one could try to develop general methods and results for comparatively simple theories, like fragments of Presburger arithmetic.

**

An interesting future line of work would be to consider other settings than arithmetic. For example, in computer science, many basic facts of functions on inductive structures seem to require non-analytic induction proofs.

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One might also approach the problem of non-analytic induction proofs from the more proof-theoretical side, for example by studying derivations in natural deduction. Dag Prawitz's (2018) recent work may be useful.

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§A Some definitions and results

DEFINITION A.1. The \mathcal{L}_0 -theory PA^- is axiomatized by the following.

- 0 and 1 are distinct:

$$0 \neq 1.$$

- Associativity of addition and multiplication:

$$(x+y)+z = x+(y+z),$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

- Commutativity of addition and multiplication:

$$x+y = y+x,$$

$$x \cdot y = y \cdot x.$$

- Distributivity of addition over multiplication:

$$x \cdot (y+z) = x \cdot y + x \cdot z.$$

- 0 is an additive identity and a multiplicative zero:

$$x+0 = x,$$

$$x \cdot 0 = 0.$$

- The order is irreflexive:

$$x \not< x.$$

- The order is transitive:

$$x < y \wedge y < z \rightarrow x < z.$$

- The order is total:

$$x < y \vee x = y \vee y < x.$$

- The order is discrete:

$$x = 0 \vee x = 1 \vee 1 < x.$$

- Addition and multiplication respect the order:

$$x < y \rightarrow x+z < y+z,$$

$$x < y \wedge 0 \neq z \rightarrow x \cdot z < y \cdot z.$$

- Smaller elements can be subtracted from larger:

$$x < y \rightarrow \exists z. x+z = y.$$

**

FACTS A.2.

- For all \mathcal{L}_0 -structures M : $M \models \text{PA}^-$ if and only if M is the non-negative part of a nontrivial discretely ordered commutative ring.
- PA^- is a theory of \mathcal{L}_0 -arithmetic.
- $\text{PA} \equiv \text{PA}^- + \{\text{IND}(\varphi) : \varphi \text{ an } \mathcal{L}_0\text{-formula in one free variable}\}$.

PROOFS. See for example Kaye's (1991) book. □

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Hetzl, Stefan and Tin Lok Wong (2017): "Some observations on the logical foundations of inductive theorem proving", Logical Methods in Computer Science 13(4).

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